

IOWA STATE UNIVERSITY

Digital Repository

Retrospective Theses and Dissertations

Iowa State University Capstones, Theses and
Dissertations

1962

Multivariate and multiple Poisson distributions

Carol Bates Edwards

Iowa State University

Follow this and additional works at: <https://lib.dr.iastate.edu/rtd>



Part of the [Mathematics Commons](#)

Recommended Citation

Edwards, Carol Bates, "Multivariate and multiple Poisson distributions " (1962). *Retrospective Theses and Dissertations*. 2127.
<https://lib.dr.iastate.edu/rtd/2127>

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

This dissertation has been 63-2965
microfilmed exactly as received

EDWARDS, Carol Bates, 1935-
MULTIVARIATE AND MULTIPLE POISSON DIS-
TRIBUTIONS.

Iowa State University of Science and Technology
Ph.D., 1962
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

MULTIVARIATE AND MULTIPLE POISSON DISTRIBUTIONS

by

Carol Bates Edwards

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of ~~Graduate~~ College

Iowa State University
Of Science and Technology
Ames, Iowa

1962

TABLE OF CONTENTS

	Page
I. INTRODUCTION	1
A. Multivariate Poisson	2
B. Multivariate Multiple Poisson	4
II. INTERCHANGE OF SUM AND LIMIT OPERATIONS	6
A. Introduction	6
B. Definitions	6
C. Chernoff's Theorem	8
D. Special Case of Chernoff's Theorem	9
E. Applications of Special Case of Chernoff's Theorem	11
III. MULTIVARIATE POISSON: MOMENT CONVERGENCE IMPLIES CONVERGENCE IN DISTRIBUTION	14
A. Introduction	14
B. Definitions	14
C. Problem of Moments	15
D. Convergence in Distribution	22
IV. MULTIVARIATE THEOREM ON FACTORIAL MOMENTS	25
A. Introduction	25
B. Definitions	25
C. Lemmas and Theorem	26
V. POISSON LIMITS OF "RUN" DISTRIBUTIONS	28
A. Introduction	28
B. Notation and Definitions	30
C. Lemmas	33
D. Theorem on "Runs"	39
E. Corollaries	42
VI. POISSON LIMITS OF CONFIGURATION DISTRIBUTIONS	44
A. Introduction	44
B. Definitions	44
C. Lemmas	46
D. Theorem on Configurations	51
E. Corollaries	53

	Page
VII. LIMIT DISTRIBUTIONS OF ARBITRARY CONFIGURATIONS WITH FIXED LENGTHS	56
A. Introduction	56
B. Notation	56
C. Procedure	57
D. Conclusions	57
VIII. THE PREVALENCE OF MULTIVARIATE POISSONS AS LIMIT DISTRIBUTIONS OF CONFIGURATIONS	59
A. Introduction	59
B. Theorem	59
IX. DERIVATION OF UNIVARIATE MULTIPLE POISSON FROM MARKOV PROCESS POSTULATES	64
A. Introduction	64
B. Assumptions	64
C. Derivation	65
X. CONCLUSIONS	68
XI. REFERENCES	72
XII. ACKNOWLEDGEMENTS	75

I. INTRODUCTION

This investigation is concerned with the derivation and description of multivariate and multivariate multiple Poisson distributions.

Three situations out of the many in which the univariate Poisson occurs (12) are reinterpreted to yield multivariate distributions.¹ Multivariate and multiple Poisson distributions arise as (1) joint distributions of linear combinations of independent Poisson random variables, (2) limit distributions of various run or configuration counts, and (3) distributions of materialization counts for particular Markov processes.

Several subsidiary results, either required or suggested by these three main investigations, are as follows: (1) generalization of a theorem of Fréchet (13) recently rediscovered by Iyer (19) which expresses factorial moments in terms of suitable probability sums, to the multivariate case; (2) demonstration that the multivariate Poisson distribution satisfies the multivariate Carleman uniqueness criterion; (3) construction of a non-independent bivariate distribution with Poisson marginals such that $E[X_1 X_2] = E[X_1]E[X_2]$, $E[X_1^2 X_2] = E[X_1^2]E[X_2]$, $E[X_1 X_2^2] = E[X_1]E[X_2^2]$; (4)

¹In this work, reference to a source listed in Chapter XI will be by the appropriate number placed in parentheses.

proof that joint distributions of arbitrary configuration counts on the circle which are asymptotically marginally Poisson, must, essentially, be asymptotically multivariate Poisson. This last essentially answers the question as to whether there exist asymptotically Poisson configuration counts that are not, jointly, asymptotically multivariate Poisson.

Distributions described here as multivariate and multivariate multiple Poisson are defined below. This will avoid any possible ambiguity (12, pp. 162, 271) due to previous use of these names to describe slightly different distributions (21), (11).

A. Multivariate Poisson

The distribution designated here as multivariate Poisson has characteristic function (c.f.)

$$c(t_1, t_2, \dots, t_m) = \exp\left\{\sum_{i=1}^m a_i z_i + \sum_{i < j} a_{ij} z_i z_j + \dots + a_{12\dots m} z_1 z_2 \dots z_m - A_m\right\} \quad (1)$$

$$\text{where } A_m = \sum_{i=1}^m a_i + \sum_{i < j} a_{ij} + \dots + a_{12\dots m}, \quad a_{j_1 j_2 \dots j_k} \geq 0,$$

$$z_j = \exp\{it_j\}.$$

This form is that used by Teicher (28, pp. 5-6) and Dwass and Teicher (11, p. 467). Though this distribution is less general than the class of linear transformations of it which

Loève (21, p. 84) calls multivariate Poisson, it is used here because it represents a logical extension of the univariate Poisson (12, p. 252), is the only infinitely divisible¹ distribution with Poisson marginals (11, p. 467), is factor closed² (28, p. 7), and arises naturally in several instances to be studied here.

Any multivariate Poisson with c.f. Equation (1) can be interpreted as the joint distribution of possibly overlapping sums of independent Poisson random variables.³ For example, let $m = 2$. Let $P(a_1)$, $P(a_2)$, $P(a_{12})$ be three independent univariate Poisson random variables with means a_1 , a_2 , and a_{12} respectively. Then the characteristic function of $[P(a_1) + P(a_{12})$, $P(a_2) + P(a_{12})]$ is $c(t_1, t_2)$ as in Equation (1).

The bivariate Poisson is of special interest in run theory and will be discussed here in some detail. It has means $a_1 + a_{12}$, $a_2 + a_{12}$, and covariance a_{12} . Its correlation coefficient given by Campbell (6, p. 20) is $r =$

¹A random variable X , its cumulative distribution functions (c.d.f.) and its characteristic function (c.f.) are called infinitely divisible if, for every positive integer p , its c.f. is the p -th power of a c.f. (21, p. 78), (11, p. 461).

²A m -variate family, \mathcal{F}_m of c.d.f.'s will be called factor closed (f.c.) if, for any c.d.f. $F \in \mathcal{F}_m$, the relationship of convolution $F = G_1 * G_2$ implies $G_1, G_2 \in \mathcal{F}_m$ (28, p. 7).

³A more general result applicable to a larger class of multivariate Poisson distributions is given by Dwass and Teicher (11, pp. 463-466).

$\left[\frac{a_{12}^2}{(a_1 + a_{12})(a_2 + a_{12})} \right]^{\frac{1}{2}}$, the usual correlation coefficient interpreted for the Poisson distribution.

When $a_{12} = 0$, the bivariate Poisson distribution is that of two independent Poissons.

When $a_1 = 0$, the bivariate Poisson is called a semi-Poisson with parameters a_2 and a_{12} . It has non-zero probability only on one-half the positive quadrant where $X_1 \leq X_2$. A similar definition holds when $a_2 = 0$.

When $a_1 = a_2 = 0$, the bivariate Poisson distribution becomes that of two equivalent Poisson random variables with means and covariance equal to a_{12} . In this case the variables are equal with probability one, so that non-zero probability occurs only on the line $X_1 = X_2$.

B. Multivariate Multiple Poisson

The m-dimensional distribution designated here as multivariate multiple Poisson is defined as the joint distribution of arbitrary sub-sums of random variables whose joint distribution is multivariate Poisson. It can also be interpreted as the joint distribution of arbitrary sub-sums of non-negative integer multiples of independent Poisson random variables.

Marginal distributions of a multivariate multiple Poisson are called univariate multiple Poissons. Each can be regarded

as a sum of non-negative integer multiples of mutually independent Poisson random variables.

The multivariate multiple Poisson is infinitely divisible, but the question of factor closure is apparently not known. Teicher (27, p. 769) states that most infinitely divisible distributions are not factor closed.

II. INTERCHANGE OF SUM AND LIMIT OPERATIONS

A. Introduction

There exists a useful theorem due to Chernoff which can be applied to the question of interchange of the operations of taking a limit and forming a joint distribution. This application will be used later to prove some convergence theorems relating to the multivariate Poisson and multivariate multiple Poisson distributions.

B. Definitions

To understand Chernoff's theorem, the following definitions are required.

Definition 2.1: Let R_m be m -dimensional Euclidean space. A function of sets E in R_m is called a distribution set function $\bar{\phi}(E)$ if it is non-negative, defined over the family of all Borel sets in R_m , $\bar{\phi}(R_m) = 1$, and countably additive, so

$$\sum_{i=1}^{\infty} \bar{\phi}(E_i) = \bar{\phi}\left(\sum_{i=1}^{\infty} E_i\right), \quad E_i E_j = \emptyset, \quad i \neq j \quad (26, \text{ p. xi}).$$

Definition 2.2: The spectrum $S(\bar{\phi})$ of a distribution set function $\bar{\phi}$ is defined as the set of all points $x = (x_1, x_2, \dots, x_m) \in R_m$ such that $\bar{\phi}(G) > 0$ for every open set G containing x (26, p.xi).

Definition 2.3: The point spectrum of $\bar{\phi}$ is the set of

all points x such that $\bar{\Phi}(x) > 0$ (26, p. xi). These points are also called discrete mass points of the distribution set function $\bar{\Phi}$ (8, p. 81).

Definition 2.4: An interval I contained in R_m is the set of points x whose coordinates satisfy conditions $a_i < x_i \leq b_i$ (26, p. xi).

Definition 2.5: Let B_1 be a Borel set in R_1 . Then the one-dimensional marginal distribution set function corresponding to the v th variable x_v , $\bar{\Phi}_v$, is defined by $\bar{\Phi}_v(B_1) = \bar{\Phi}(B_1 \times J)$ where $B_1 \times J$ is the m -dimensional cylinder set with base B_1 (8, p. 82).

Definition 2.6: An interval I is a continuity interval of $\bar{\Phi}$ if none of the extremes a_v and b_v is a discrete mass point for the one-dimensional distribution set function $\bar{\Phi}_v$ (8, p. 80).

Definition 2.7: A sequence of distribution set functions $\{\bar{\Phi}_n(E)\}$ converges in distribution to $\bar{\Phi}(E)$ if $\lim_{n \rightarrow \infty} \bar{\Phi}_n(I) = \bar{\Phi}(I)$ for all continuity intervals I of the distribution set function $\bar{\Phi}$ (8, pp. 82-83). Write $\bar{\Phi}_n \rightarrow \bar{\Phi}$. Occasionally it will be convenient to refer to a "random variable" converging in distribution to $\bar{\Phi}$. In such cases the symbol \xrightarrow{d} will be used.

Definition 2.8: A real-valued function f defined on R_m to R_q is Borel measurable if for every $c = (c_1, c_2, \dots, c_q)$ the set $\{x \in R_m \mid f(x) \leq c\}$ is a Borel set (8, p. 37).

Definition 2.9: Let $D(f)$ be the set of points of discontinuity of f .

Definition 2.10: Let $\bar{\Phi}$ be a distribution set function defined on R_m and f a Borel measurable function from R_m to R_q . Let $D(f)$ have $\bar{\Phi}$ measure zero. For any Borel set $B_y \subseteq R_q$ let $B_x = \{x \in R_m \mid f(x) \in B_y\}$, a Borel set $\subseteq R_m$. Define the $\bar{\Phi}$ induced distribution set function ψ on all Borel sets $B_y \subseteq R_q$ as $\psi(B_y) = \bar{\Phi}(B_x)$ (16, p. III-23).

Definition 2.11: Define the characteristic function $c(t) = c(t_1, t_2, \dots, t_m)$ for the distribution set function $\bar{\Phi}$ on R_m as $\int_{R_m} \exp\{itx\} d\bar{\Phi}$ where t and $x \in R_m$ (8, pp. 100-101).

C. Chernoff's Theorem

Theorem 2.1 (Chernoff, 7, p. 8): Let $\{\bar{\Phi}_n\}$ denote a sequence of distribution set functions on R_m . Let f be a Borel measurable function from R_m to R_q . Let ψ_n be the $\bar{\Phi}_n$ induced distribution set function defined on R_q . If $\bar{\Phi}_n \rightarrow \bar{\Phi}$ and $D(f)$ has $\bar{\Phi}$ measure zero, then $\psi_n \rightarrow \psi$, where ψ is the $\bar{\Phi}$ induced distribution set function defined on R_q .

In other words, Chernoff states that when a sequence of distribution set functions $\{\bar{\Phi}_n\}$ converges to $\bar{\Phi}$, then for any Borel measurable function f whose points of discontinuity have $\bar{\Phi}$ measure zero, the corresponding sequence of induced distribution set functions $\{\psi_n\}$ converges to the $\bar{\Phi}$ induced distribution set function ψ .

bution set function ψ .

D. Special Case of Chernoff's Theorem

A special case of Theorem 2.1 obtains when the function $f(x_1, x_2, \dots, x_m)$ is a vector in R_q whose components are possibly overlapping sums of the arguments of $f(x)$. It is possible to give an independent proof of this special case by appeal to Cramér's multivariate continuity theorem, namely:

Theorem 2.2 (Cramér, 8, pp. 102-103): Let a sequence of distribution set functions $\{\bar{\phi}_n\}$, $\bar{\phi}_n$ defined on R_m , and the corresponding sequence of characteristic functions $\{c_n(t)\}$ be given. A necessary and sufficient condition for $\bar{\phi}_n \rightarrow \bar{\phi}$ is that, for every t , the sequence $\{c_n(t)\}$ converges to a limit $c(t)$, which is continuous at the special point $t = 0$.

When this condition is satisfied the limit $c(t)$ is identical with the characteristic function of the limit distribution set function. The special case of Theorem 2.1 will be proved as follows:

Proof:

1. Let $f(x) = (f_1, f_2, \dots, f_q) \in R_q$ where f_α , $\alpha = 1, 2, \dots, q$, is the α -th sub-sum of arguments of $x \in R_m$.
2. f is Borel measurable since $\{x \in R_m \mid f(x) \leq c\}$ is the intersection of half spaces and therefore a Borel set in R_q for any $c \in R_q$.
3. $D(f)$ has measure 0 since, in view of the continuity

of f , $D(f)$ is the null set.

4. Therefore there exist $\bar{\phi}_n$ and $\bar{\phi}$ induced distribution set functions ψ_n and ψ , respectively, on R_q , by Definition 2.10.

5. Let $c_n(t)$, $c(t)$, $d_n(s)$, $d(s)$ be the characteristic functions of $\bar{\phi}_n$, $\bar{\phi}$, ψ_n , ψ respectively. Since $\bar{\phi}_n \rightarrow \bar{\phi}$, then $c_n(t) \rightarrow c(t)$ where $c(t)$ is continuous at $t = 0$, by Theorem 2.2 above.

6. In order to show that $\psi_n \rightarrow \psi$ show that $\lim_{n \rightarrow \infty} d_n(s) = d(s)$ and apply Theorem 2.2 again.

$$\begin{aligned}
 7. \quad \lim_{n \rightarrow \infty} d_n(s) &= \lim_{n \rightarrow \infty} \int_{R_q} \exp\{i \sum_{\alpha=1}^q s_{\alpha} f_{\alpha}\} d\psi_n \\
 &= \lim_{n \rightarrow \infty} \int_{R_m} \exp\{i \sum_{k=1}^m x_k(\Sigma s)_k\} d\bar{\phi}_n \\
 &= \lim_{n \rightarrow \infty} c_n[(\Sigma s)] = c[(\Sigma s)]
 \end{aligned}$$

where the middle step holds for every n since the integrands can be approximated by simple functions, and the continuity of $c[(\Sigma s)]$ at the origin of R_q follows from the continuity of $c(s)$ at the origin of R_m .

$$\begin{aligned}
 8. \quad \text{However } d(s) &= \int_{R_q} \exp\{i \sum_{\alpha=1}^q s_{\alpha} f_{\alpha}\} d\psi \\
 &= \int_{R_m} \exp\{i \sum_{k=1}^m x_k(\Sigma s)_k\} d\bar{\phi} = c[(\Sigma s)]
 \end{aligned}$$

by similar reasoning. Therefore, $d_n(s) \rightarrow d(s)$ which is continuous at $s = 0$, and $\psi_n \rightarrow \psi$.

Q.E.D.

E. Applications of Special Case of Chernoff's Theorem

In this section convergence properties of the multivariate Poisson and multivariate multiple Poisson distributions will be investigated by means of special case of Theorem 2.1. For this study, knowledge of several characteristics of these Poisson distributions is important.

Characteristic 2.1: Let $\bar{\phi}$ be the distribution set function on R_m corresponding to the m -variate independent Poisson distribution with parameters λ_1 . Then the univariate distribution set function ψ corresponding to $f(x_1, x_2, \dots, x_m) = x_1 + x_2 + \dots + x_m$ is that corresponding to the univariate Poisson distribution with parameter $\sum_{i=1}^m \lambda_i$.

More simply stated, if X_1, X_2, \dots, X_m are univariate independent Poissons with parameters λ_i , then $\sum_{i=1}^m X_i$ is univariate Poisson with parameter $\sum_{i=1}^m \lambda_i$.

Characteristic 2.2: If X_1, X_2, \dots, X_m are multivariate independent Poisson, then subsets of them, suitably summed, are multivariate Poisson with c.f. Equation (1).

Characteristic 2.3: If X_1, X_2, \dots, X_m are multivariate

Poisson then subsets of them, suitably summed, are multivariate multiple Poisson.

Then, by application of the special case of Theorem 2.1, the following results hold.

Corollary 2.1: If $X_1^{(n)}, X_2^{(n)}, \dots, X_m^{(n)} \xrightarrow{d}$ a multivariate independent Poisson with parameters λ_i , then $\sum_{i=1}^m X_i^{(n)} \xrightarrow{d}$ a univariate Poisson with parameter $\sum_{i=1}^m \lambda_i$.

Corollary 2.2: If $X_1^{(n)}, X_2^{(n)}, \dots, X_m^{(n)} \xrightarrow{d}$ a multivariate independent Poisson, then subsets, suitably summed \xrightarrow{d} a multivariate Poisson.

Corollary 2.3: If $X_1^{(n)}, X_2^{(n)}, \dots, X_m^{(n)} \xrightarrow{d}$ a multivariate Poisson, then subsets of them, suitably summed, \xrightarrow{d} a multivariate multiple Poisson.

A particular instance which illustrates the usefulness of these corollaries is a derivation of the correlated bivariate Poisson as the limit of the multinomial distribution alternate to that of and simpler than Teicher's (28, p. 1). The simplicity accrues precisely because of inversion of the order of taking the limit and the computation of the induced distribution.

Teicher derives the correlated bivariate Poisson by considering the multinomial distribution of numbers of simultaneous occurrence or non-occurrence of two events in n trials,

X_{11} , X_{10} , X_{01} , X_{00} . He finds the bivariate distribution of $(X_{11} + X_{10}, X_{11} + X_{01})$ and shows in the limit it is the distribution termed correlated Poisson by Campbell (6, pp. 19-20) and Aitken (1, pp. 94-95), who derive it via generating functions.

A simpler way to reach this conclusion is to consider the limit of the joint distribution of X_{11} , X_{10} , X_{01} , X_{00} which, under the conditions that n is large and $np_{11} = \mu_1$, $np_{10} = \mu_2$, $np_{01} = \mu_3$ are fixed, is that of three independent Poissons (12, p. 167). Application of Corollary 2.2 states that the bivariate distribution of $(X_{11} + X_{10}, X_{11} + X_{01})$ is bivariate Poisson. The particular correlated Poisson can be found by means of characteristic functions as in the proof of the special case of Theorem 2.1. That is, since the characteristic function of X_{11} , X_{10} , X_{01} in the limit is $c(t_1, t_2, t_3) =$

$\exp\{\sum_{j=1}^3 \mu_j(z_j - 1)\}$, the characteristic function of $(X_{11} + X_{10}, X_{11} + X_{01})$ is $d(s_1, s_2) = c(s_1, s_2, s_1 + s_2) = \exp\{\mu_1(z_1 - 1) + \mu_2(z_2 - 1) + \mu_3(z_1 z_2 - 1)\}$ where $z_j = \exp\{is_j\}$.

III. MULTIVARIATE POISSON: MOMENT CONVERGENCE IMPLIES CONVERGENCE IN DISTRIBUTION

A. Introduction

It will be shown below that the moments of a multivariate Poisson specify its distribution uniquely. and thus, by Haviland (18, p. 632), that any sequence of distribution functions whose moments converge to those of the multivariate Poisson converges in distribution to the multivariate Poisson.

B. Definitions

In order to discuss the problem of moments in some generality the following definitions additional to those of Section II. B are given.

Definition 3.1: Two distribution set functions are said to be substantially equal if they have the same intervals of continuity and their values coincide over all such intervals (26, p. xii).

Definition 3.2: The $v = (v_1, v_2, \dots, v_m)$ -th moment about the origin for the distribution set function $\bar{\phi}$ is defined as

$$\mu'_{v_1, v_2, \dots, v_m} = \int_{R_m} x_1^{v_1} x_2^{v_2} \cdots x_m^{v_m} d\bar{\phi}, \quad v_1, v_2, \dots, v_m = 0, 1, 2, \dots,$$

where a Lebesgue-Stieltjes integral is assumed (8, pp. 62-63, 85).

C. Problem of Moments

The general problem of moments has two aspects, existence and uniqueness of a solution. The problem can be expressed precisely as follows (26, p. 1).

Let R_m be m -dimensional Euclidean space. Let an infinite multiple sequence of real constants $\mu'_{v_1, v_2, \dots, v_m}$, v_1, v_2, \dots , $v_m = 0, 1, 2, \dots$, be given. Find necessary and sufficient conditions that there exist a m -dimensional distribution set function Φ whose spectrum $S(\Phi)$ is contained in a closed set S_0 , given in advance, and which satisfies

$$\mu'_{v_1, v_2, \dots, v_m} = \int_{R_m} x_1^{v_1} x_2^{v_2} \dots x_m^{v_m} d\Phi.$$

If the moment problem has a solution which is substantially unique (all solutions substantially equal) the moment problem is said to be determined. Otherwise it is called indeterminate. The moment problem is always determined if a solution exists whose spectrum is a bounded set (26, p. 11). However, when the spectrum is unbounded there may be more than one solution to the problem of moments. Kendall (20, pp. 105-106) gives several examples.

Two well-known specialized moment problems are the Hamburger moment problem, where S_0 coincides with the real axis, and the Stieltjes moment problem, where S_0 coincides with the non-negative part of the real axis (26, p. 4). It is possible

that the Hamburger moment problem may be indeterminate while the corresponding Stieltjes moment problem with the same $\mu'_{v_1, v_2, \dots, v_m}$ is determined (26, p. x). In other words, given a certain sequence of moments it is possible that there exists only one distribution set function defined on the non-negative real line that possesses these moments, but that there exists more than one distribution set function defined on the whole line possessing these same moments. However, if a sequence of moments is such that the Hamburger moment problem is determined, then the Stieltjes moment problem must be determined too. Indeed, a generalized Hamburger moment criterion will be used below to determine uniqueness of a distribution set function whose spectrum is known to occupy only the non-negative portion of m -dimensional space.

The multivariate Poisson distribution is determined uniquely by its moments. To show this only the second aspect of the problem of moments need be considered, namely, uniqueness of the distribution, since the first aspect, existence of at least one distribution with multivariate Poisson moments is satisfied by the multivariate Poisson distribution itself.

The proof of uniqueness is based on the following theorem which generalizes a sufficient condition proposed by Carleman for determinacy of the Hamburger moment problem. Call this extended criterion the multivariate Hamburger Carleman criterion.

Theorem 3.1 (Cramér and Wold, 9, pp. 291-292): Let the m -dimensional moment problem corresponding to the moments $\mu'_{v_1, v_2, \dots, v_m}$, $v_1, v_2, \dots, v_m = 0, 1, 2, \dots$, have a solution. Let $\lambda_{2k} = \mu'_{2k, 0, 0, \dots, 0} + \mu'_{0, 2k, 0, \dots, 0} + \dots + \mu'_{0, 0, 0, \dots, 2k}$. A sufficient condition for the moment problem to be determined is that $\sum_{k=1}^{\infty} \lambda_{2k}^{-1/2k}$ diverges.

Though this theorem was first proved by Cramér and Wold, the statement above is substantially that of Shohat and Tamarkin (26, p. 21).

It will now be shown that the multivariate Poisson moments satisfy this criterion.

Theorem 3.2: The moments of the multivariate Poisson distribution, say $\mu'_{v_1, v_2, \dots, v_m}$, $v_1, v_2, \dots, v_m = 0, 1, 2, \dots$, satisfy the multivariate Hamburger Carleman criterion, namely

that $\sum_{k=1}^{\infty} \lambda_{2k}^{-1/2k}$ diverges where $\lambda_{2k} = \mu'_{2k, 0, 0, \dots, 0} + \mu'_{0, 2k, 0, \dots, 0} + \dots + \mu'_{0, 0, 0, \dots, 2k}$.

Proof:

1. Let $\mu'_{0, 0, \dots, k, \dots, 0} = \mu'_k$ for simplicity.
2. The k -th factorial moment for the marginal Poisson with parameter a_j is

$$\mu'_k = \sum_{x=k}^{\infty} \frac{x!}{(x-k)!} \frac{e^{-a_j} (a_j)^x}{x!} = a_j^k \sum_{x=k}^{\infty} \frac{e^{-a_j} (a_j)^{x-k}}{(x-k)!} = a_j^k.$$

3. By definition $\mu'_k = k! p(k) + \sum_{r=k+1}^{\infty} \frac{r!}{(r-k)!} p(r)$

where $p(r) = \frac{e^{-a_j} (a_j)^r}{r!}$.

4. By means of Stirling's approximations (12, pp. 50-52), namely,

$$(2\pi)^{\frac{1}{2}} n^{\frac{n+1}{2}} e^{-n+\frac{1}{12n+1}} < n! < (2\pi)^{\frac{1}{2}} n^{\frac{n+1}{2}} e^{-n+\frac{1}{12n}},$$

$$n! \doteq (2\pi)^{\frac{1}{2}} n^{\frac{n+1}{2}} e^{-n}$$

one has $\frac{r!}{(r-k)!} > \left[\frac{r}{r-k} \right]^{r-k+\frac{1}{2}} r^k e^{-k} e^{\frac{1}{12r+1} - \frac{1}{12(r-k)}}$.

5. Since $\left[\frac{r}{r-k} \right] > 1$ and $-\frac{1}{12} < \frac{1}{12r+1} - \frac{1}{12(r-k)} < 0$ for

$r \geq k+1 > 0$, one has $\frac{r!}{(r-k)!} > r^k e^{-k} e^{-\frac{1}{12}}$.

$$\begin{aligned} 6. \text{ Hence } \mu'_k &= a_j^k > k! p(k) + e^{-\frac{1}{12}} e^{-k} \sum_{r=k+1}^{\infty} r^k p(r) \\ &= e^{-a_j} (a_j)^k + e^{-\frac{1}{12}} e^{-k} \left[\mu'_k - \sum_{r=0}^k r^k p(r) \right] \\ &> e^{-a_j} a_j^k + e^{-\frac{1}{12}} e^{-k} [\mu'_k - k^k]. \end{aligned}$$

7. This can be written as

$$\mu'_k < a_j^k e^{\frac{1}{12}} e^k (1 - e^{-a_j}) + k^k < a_j^k e^{\frac{1}{12}} e^k + k^k.$$

8. For $k > K$, where K is chosen so $K \geq a_j e^{1+\frac{1}{12k}}$, one has $\mu'_k < 2k^k$.

9. Therefore, by definition of λ_{2k} and the inequality above, $\lambda_{2k} < 2m(2k)^{2k}$ for k sufficiently large.

10. Then $\lambda_{2k}^{-1/2k} \geq \frac{1}{(2m)^{2k} 2k} \geq \frac{1}{(2m)(2k)}$ since $(2m)^{1/2k} < 2m$.

11. Hence $\sum_{k=1}^{\infty} \lambda_{2k}^{-1/2k} \geq \frac{1}{4m} \sum_{k=1}^{\infty} \frac{1}{k}$ which diverges.

Q.E.D.

It is to be noted that Theorems 3.1 and 3.2 together imply that there exists substantially no other distribution besides the multivariate Poisson itself which possesses multivariate Poisson moments.

A further remark concerning uniqueness is as follows. It is an open question whether for arbitrarily large k there exist two substantially unequal distributions with the same infinite set of marginal Poisson moments and with the same finite set of Poisson cross moments of degree¹ $\leq k$. An example given below shows that two such substantially unequal distributions do exist in the case $k = 3$. It is of independent interest that this example also shows that two random

¹A moment $\mu'_{v_1, v_2, \dots, v_m}$ has degree k if $\sum_{i=1}^m v_i = k$.

variables can have zero correlation¹ of degree 3 and be marginally Poisson, yet their joint distribution need not be Poisson nor independent. Anderson (2, pp. 37-38) has shown the same is true for the normal distribution though he only concerns himself with zero correlation of degree 2.

The example is as follows. Consider two random variables X_1, X_2 whose joint distribution is a correlated bivariate Poisson with p.g.f.

$$g(z_1, z_2) = \exp\{a_1(z_1-1) + a_2(z_2-1) + a_{12}(z_1 z_2 - 1)\}$$

where $a_1, a_2, a_{12} > 0$, $a_1 = a_2 = a$, $2(a + a_{12}) =$ a positive integer. Then, by a familiar device (2, p. 37), construct a second distribution from the one above by altering the probabilities of the four events $(0,0)$, $(\Delta,0)$, $(0,\Delta)$, (Δ,Δ) by adding them to the constants δ , $-\delta$, $-\delta$, δ respectively, where Δ is a positive integer and δ a constant such that it does not give negative probabilities.

Let $\left|_i\right.$ mean the i -th distribution under discussion. Then the following relations exist between the cross and marginal moments of the above two distributions:

$$\mathbb{E}\left|_2 X_1 X_2 = \mathbb{E}\left|_1 X_1 X_2 + \Delta^2 \delta = \mathbb{E}\left|_1 X_1 \mathbb{E}\left|_1 X_2 + a_{12} + \Delta^2 \delta,\right.\right.\right.$$

¹Zero correlation of degree k means $\mu_{v_1, v_2, \dots, v_m}^i = \mu_{v_1, 0, \dots, 0}^i \mu_{0, v_2, \dots, 0}^i \dots \mu_{0, 0, \dots, v_m}^i$ for all moments of degree $\leq k$.

$$\begin{aligned}
E\big|_2 X_1^2 X_2 &= E\big|_1 X_1^2 X_2 + \Delta^3 \delta \\
&= E\big|_1 X_1^2 E\big|_1 X_2 + 2a_{12}(a+a_{12}) + a_{12} + \Delta^3 \delta = E\big|_2 X_1 X_2^2, \\
E\big|_2 X_1^2 X_2^2 &= E\big|_1 X_1^2 X_2^2 + \Delta^4 \delta \\
&= E\big|_1 X_1^2 E\big|_1 X_2^2 + 4a_{12}(a+a_{12})^2 + 4a_{12}(a+a_{12}) + 2a_{12}^2 \\
&\quad + a_{12} + \Delta^4 \delta.
\end{aligned}$$

Choose Δ and δ so that the second distribution has zero correlation of order 3, namely $\Delta = 2(a+a_{12}) + 1$, $\delta = -\frac{a_{12}}{\Delta^2}$. In

that case $E\big|_2 X_1^2 X_2^2 = E\big|_1 X_1^2 E\big|_1 X_2^2 + 2a_{12}^2$ and the second distribution is not the joint distribution of two independent random variables.

Further, this second distribution is not Poisson, since if it were then by Teicher (28, p. 8) the $2^p - 1 = 3$ mixed moments of order $\leq p = 2$ (and no more than the first power in any variable) would completely specify it. However the three

moments $\mu'_{10} = E\big|_1 X_1$, $\mu'_{01} = E\big|_1 X_2$, $\mu'_{11} = E\big|_2 X_1 X_2$ are those of the joint distribution of two independent Poisson random variables. Since it is known already that this second distribution is not the distribution of two independent random variables, it cannot be Poisson.

D. Convergence in Distribution

In the body of this dissertation a sequence of moments $\{\mu'_{v_1, v_2, \dots, v_m} | \phi_n\}$ is considered which corresponds to a sequence of distribution functions $\{\phi_n\}$. When these moments converge to those of the multivariate Poisson, the question is whether or not the latter distribution is the limit of the sequence of distribution functions $\{\phi_n\}$. Haviland, in a theorem similar to the continuity theorem for characteristic functions in R_m , has given sufficient conditions under which convergence of moments implies convergence in distribution.

Theorem 3.3 (Haviland, 18, p. 632): Let $\{\phi_n(E)\}$ be a given distribution set functions defined on R_m such that:

$$1. \quad \mu'_{v_1, v_2, \dots, v_m} | \phi_n = \int_{E \in R_m} x_1^{v_1} x_2^{v_2} \dots x_m^{v_m} d\phi_n(E)$$

exists for $n = 1, 2, \dots$ (or at least from a certain rank n on, possibly depending on v_1, v_2, \dots, v_m);

2. for any fixed set v_1, v_2, \dots, v_m , the moments $\mu'_{v_1, v_2, \dots, v_m} | \phi_n$ lie, when they exist, between two fixed limits independent of n (but possibly dependent on v_1, v_2, \dots, v_m);

$$3. \quad \lim_{n \rightarrow \infty} \mu'_{v_1, v_2, \dots, v_m} | \phi_n \text{ exists, say } = \mu'_{v_1, v_2, \dots, v_m}$$

for all non-negative values of v_1, v_2, \dots, v_m .

Then there exists at least one distribution function, say $\bar{\phi}$, such that $\mu'_{v_1, v_2, \dots, v_m} = \mu'_{v_1, v_2, \dots, v_m} \big|_{\bar{\phi}}$ and a subsequence $\{\bar{\phi}_{n_i}\}$ can be extracted from the given sequence of distribution set functions so that $\lim_{i \rightarrow \infty} \bar{\phi}_{n_i}(I) = \bar{\phi}(I)$ for all continuity intervals I of $\bar{\phi}$.

If, in addition, the sequence $\{\mu'_{v_1, v_2, \dots, v_m}\}$ is such that $\bar{\phi}$ is uniquely determined by it, then the sequence $\{\bar{\phi}_n\}$ itself converges as n becomes infinite to $\bar{\phi}$ on any continuity interval I of $\bar{\phi}$.

This leads to the following theorem.

Theorem 3.4: Let $\lambda_{v_1, v_2, \dots, v_m}$, $v_1, v_2, \dots, v_m = 0, 1, 2, \dots$ be the moments of a multivariate Poisson distribution ϕ_p . Consider the sequence of distribution set functions $\{\bar{\phi}_n\}$ defined on R_m with moments $\mu'_{v_1, v_2, \dots, v_m} \big|_{\bar{\phi}_n}$. If these moments satisfy conditions 1 and 2 in Theorem 3.3, and if $\lim_{n \rightarrow \infty} \mu'_{v_1, v_2, \dots, v_m} \big|_{\bar{\phi}_n} = \lambda_{v_1, v_2, \dots, v_m}$ then $\lim_{n \rightarrow \infty} \bar{\phi}_n(I) = \phi_p(I)$ for all continuity intervals I of ϕ_p .

Proof:

1. The moments $\lambda_{v_1, v_2, \dots, v_m}$, $v_1, v_2, \dots, v_m = 0, 1, 2, \dots$, determine the multivariate Poisson distribution uniquely by Theorems 3.1 and 3.2.

2. $\lim_{n \rightarrow \infty} \mu'_{v_1, v_2, \dots, v_m} \big|_{\bar{\phi}_n}$ exists, for given it equals

$\lambda_{v_1, v_2, \dots, v_m}$.

3. Therefore $\lim_{n \rightarrow \infty} \phi_n(I) = \phi_p(I)$ for all continuity intervals of ϕ_p , by Theorem 3.3.

Q.E.D.

IV. MULTIVARIATE THEOREM ON FACTORIAL MOMENTS

A. Introduction

This theorem will be used to obtain the factorial moments for the joint distribution of numbers of runs of different types. It is the key to the derivation here of the multivariate Poisson. In the univariate case for runs this relation has been derived and used by von Mises (22). Fréchet (13) also proves this relation in the univariate case, but for a more abstract situation. Iyer (19) rediscovers this univariate relation and applies it not only to moments of distributions arising from a sequence of observations belonging to a binomial population but also to those belonging to two continuous populations. His proof by induction can be generalized to give the multivariate theorem; however, a shorter proof suggested by the univariate arguments of Fréchet and von Mises is presented here.

B. Definitions

Definition 4.1: Let $\prod_1 X_i = [X]$.

Definition 4.2: Consider a finite set Ω of events divided in some fashion into k subsets Ω_i containing respectively N_i events. Let ω_i be a particular subset of Ω_i containing $n(\omega_i)$ events. Let A denote the materialization of the $\sum_i n(\omega_i)$

events in $\bigcup_i \omega_i$ and let B denote the non-materialization of the $\sum_i (N_i - n(\omega_i))$ events in $\bigcup_i (\Omega_i - \omega_i)$. Then

$p(\omega_1, \omega_2, \dots, \omega_k) \equiv \Pr\{A \wedge B\}$, and $P(\omega_1, \omega_2, \dots, \omega_k) \equiv \Pr\{A\}$.

Definition 4.3: $V(x) \equiv V(x_1, x_2, \dots, x_k)$ is defined to be the class of distinct set vectors $(\omega_1, \omega_2, \dots, \omega_k)$ that can be formed under the restriction $n(\omega_i) = x_i$, $i = 1, 2, \dots, k$.

Definition 4.4: Let $s(n) \equiv s(n_1, n_2, \dots, n_k) \equiv \sum p(\omega_1, \omega_2, \dots, \omega_k)$ where the summation extends over $V(n)$.

Definition 4.5: Let $S(v) \equiv S(v_1, v_2, \dots, v_k) \equiv \sum p(\omega_1, \omega_2, \dots, \omega_k)$ where the summation extends over $V(v)$.

Definition 4.6: Let $I \equiv (I_1, I_2, \dots, I_k)$ be the k-dimensional chance variable whose i-th component equals the number of elements of Ω_i that materialize.

Definition 4.7: Let $K \equiv \sum p(\omega_1, \omega_2, \dots, \omega_k) \left[\binom{n(\omega)}{v} \right]$

where the summation extends over all the $\sum_{n=v}^N \left[\binom{N}{n} \right]$ distinct p's with $v \leq n \leq N$.

C. Lemmas and Theorem

Lemma 4.1: $\Pr\{I = n\} = s(n)$.

Proof:

It follows from Definitions 4.4 and 4.6.

Q.E.D.

Lemma 4.2: $S(v) = K$.

Proof:

Consider each of the $\left[\binom{N}{v} \right]$ terms P of $S(v)$ expanded

into a sum of $\left[2^{N-v} \right]$ terms p . The resulting series of $\left[\left(\frac{N}{v} \right) 2^{N-v} \right]$ terms for $S(v)$ contains all terms p with $v \leq n \leq N$, and only such terms, with any particular $p(\omega_1, \omega_2, \dots, \omega_k)$ appearing exactly $\left[\binom{n(v)}{v} \right]$ times.

Q.E.D.

Note, parenthetically, that the series for $S(v)$ contains $\left[\left(\frac{N}{v} \right) 2^{N-v} \right]$ terms p , while the series for K contains $\sum_{n=v}^N \left[\binom{n}{v} \binom{N}{n} \right]$ terms, and indeed, $\sum_{n=v}^N \left[\binom{n}{v} \binom{N}{n} \right] = \left[\sum_{n=v}^N \binom{n}{v} \binom{N}{n} \right] = \left[\left(\frac{N}{v} \right) \sum_{n=v}^N \binom{N-v}{n-v} \right] = \left[\left(\frac{N}{v} \right) 2^{N-v} \right]$.

Lemma 4.3: $\sum_{n=v}^N \left[\binom{n}{v} \right] s(n) = K$.

Proof:

Consider the terms being summed in K . Summing these first over p for fixed n yields $s(n)$. Summing next over n yields the left-hand side.

Q.E.D.

Theorem 4.1: Let $\mu(v) = \mu(v_1, v_2, \dots, v_k)$ be the factorial moment of order v of I . Then $S(v) \left[v! \right] = \mu(v)$.

Proof:

By Lemmas 4.1, 4.2, and 4.3 $S(v) = \sum_{n=v}^N \left[\binom{n}{v} \right] \Pr\{I = n\}$

$$= \frac{E \left[n \left[v \right] \right]}{\left[v! \right]} \text{ where } n \left[v \right] = \frac{n!}{(n-v+1)!}.$$

Q.E.D.

V. POISSON LIMITS OF "RUN" DISTRIBUTIONS

A. Introduction

Many people have discussed run theory. Mood (23, p. 367) dates the origin of the theory toward the end of the nineteenth century and gives a brief historical review. The first correct derivation of the mean and variance of runs from a binomial population was that of von Bortkiewicz (5) in 1917. Shortly thereafter, von Mises (22) showed that the number of runs is approximately distributed according to the Poisson law for large samples under the condition of constant expectation. In 1926 Wishart and Hirschfeld (31) showed that the distribution of the total number of runs without regard to length in samples from a binomial population is asymptotically normal. In 1940 Wald and Wolfowitz (30) showed that the distribution of the total number of runs (again without regard to length) from arrangements of a fixed number of two kinds of elements is asymptotically normal. Mood (23), also in 1940, showed that when the variables -- numbers of runs of various given lengths of elements from either random arrangements of a fixed number of elements of two or more kinds or from binomial and multinomial populations -- are standardized, their joint distribution is asymptotically normal as sample size increases. Mosteller (24) has applied run theory to quality control.

Mood's work can be construed as wrapping up the normal

theory of runs when the dichotomy criterion (the criterion by which a ball is judged black or white) is either a sample quantile (the case of random arrangements of a fixed number of elements of two or more kinds) or an infinite population quantile (in the case of sampling from a binomial population). David (10) has considered several additional dichotomy criteria including the sample mean when sampling from a normal population. However the present work has concentrated on sampling from a binomial population, and it will be shown below that under certain conditions the joint distribution of numbers of runs from a binomial population is multivariate independent Poisson.

Consider a circle with n positions, each of which can be filled by a white ball, o , with probability $p(n)$, or by a black ball, x , with probability $q(n)$, $p(n) + q(n) = 1$. A "run" is defined as a succession of events of one kind preceded and succeeded by events of a second kind (23, p. 367). Let $I[k(n), n]$ be the number of "runs" of length $k(n) \geq 1$ of white balls on a circle with n positions, and let $J[\ell(n), n]$ be the number of "runs" of length $\ell(n) \geq 1$ of black balls on a circle with n positions. It will be shown that if the $r+s$ expectations, $E\{I[k_1(n), n]\}$ and $E\{J[\ell_j(n), n]\}$, $k_1(n) \neq k_{1'}(n)$ and $\ell_j(n) \neq \ell_{j'}(n)$, all tend to positive constants with increasing n , the $r+s$ random variables $I[k_1(n), n]$, $I[k_2(n), n]$, \dots , $I[k_r(n), n]$, $J[\ell_1(n), n]$, $J[\ell_2(n), n]$,

$\dots, J[\ell_s(n), n]$ are asymptotically mutually independent Poisson. Univariate asymptotic Poissonness was established by von Mises (2¹) for the random variable $I[k(n), n] + J[k(n), n]$.

The derivation here of the asymptotic joint distribution of the $r+s$ random variables $I[k_i(n), n], J[\ell_j(n), n], i = 1, 2, \dots, r, j = 1, 2, \dots, s$, involves essentially identification of the factorial moments in the limit. To compute the factorial moments, Theorem 4.1 is used with the following interpretation of its notation.

B. Notation and Definitions

Let α be a subscript which, for $\alpha = 1, 2, \dots, r$, enumerates white ball run types, and, for $\alpha = r+1, r+2, \dots, r+s$, enumerates black ball run types. $\Omega_\alpha, \alpha = 1, 2, \dots, r$, consists of the $N_\alpha = n$ events that a run of $k_\alpha(n)$ white balls, $k_\alpha(n) \neq k_{\alpha'}(n)$, starts at position t on the circle, $t = 1, 2, \dots, n$; $\Omega_\alpha, \alpha = r+1, r+2, \dots, r+s$, consists of the $N_\alpha = n$ events that a run of $\ell_\alpha(n)$ black balls, $\ell_\alpha(n) \neq \ell_{\alpha'}(n)$, starts at position t . ω_α , a particular subset of Ω_α , contains $n(\omega_\alpha)$ events corresponding to $n(\omega_\alpha)$ distinct positions.

A is the simultaneous starting of $n(\omega_\alpha)$ runs of type α at $n(\omega_\alpha)$ positions on the circle, $\alpha = 1, 2, \dots, r+s$. The set of $n(\omega_\alpha)$ distinct positions may overlap the set of $n(\omega_{\alpha'})$ distinct positions for $\alpha \neq \alpha'$.

B is the simultaneous failure of a run of type α to start at any of the $n - n(\omega_\alpha)$ remaining positions on the circle, $\alpha = 1, 2, \dots, r+s$.

$p(\omega_1, \omega_2, \dots, \omega_{r+s})$ is the joint probability that exactly $n(\omega_\alpha)$ runs of type α materialize, and that they start at precisely the $n(\omega_\alpha)$ positions on the circle specified by ω_α , $\alpha = 1, 2, \dots, r+s$. Similarly, $P(\omega_1, \omega_2, \dots, \omega_{r+s})$ is the joint probability that at least $n(\omega_\alpha)$ runs of type α materialize, and that the $n(\omega_\alpha)$ positions specified by ω_α be included among the starting points, $\alpha = 1, 2, \dots, r+s$.

$V(x_1, x_2, \dots, x_{r+s})$ is the set of $\prod_{\alpha=1}^{r+s} \binom{n}{x_\alpha}$ possible selections of different though possibly overlapping subsets ω_α from Ω_α , under the restriction $n(\omega_\alpha) = x_\alpha$, $\alpha = 1, 2, \dots, r+s$. Essentially, $V(x)$ lists all possibilities for starting x_α runs of type α on the circle, $\alpha = 1, 2, \dots, r+s$.

Definition 5.1: Define the $r+s$ -dimensional random variable $I(n) = (I_1(n), I_2(n), \dots, I_{r+s}(n))$ whose α -th component $I_\alpha(n)$ equals the number of runs of type α that materialize on the circle. In other words,

$$I_\alpha(n) = \begin{cases} I[k_\alpha(n), n] & \alpha = 1, 2, \dots, r, k_\alpha(n) \geq 1 \\ J[l_\alpha(n), n] & \alpha = r+1, r+2, \dots, r+s, l_\alpha(n) \geq 1 \end{cases}$$

where $k_\alpha(n) \neq k_{\alpha_1}(n)$ and $l_\alpha(n) \neq l_{\alpha_1}(n)$, and either r or s but not both may be equal to zero, signifying the absence of runs

of the corresponding color.

Note that for some values of $(w_1, w_2, \dots, w_{r+s})$ the corresponding $P(w_1, w_2, \dots, w_{r+s}) = 0$, because of the impossibility of arranging runs of the required type in the positions specified by $(w_1, w_2, \dots, w_{r+s})$. Non-zero probabilities occur either when the vector $(w_1, w_2, \dots, w_{r+s})$ specifies an arrangement of

$\sum_{\alpha=1}^{r+s} v_{\alpha}$ runs with no overlap (join) or when $(w_1, w_2, \dots, w_{r+s})$

specifies an arrangement of this many runs with overlap (join) of the following types: (1) overlap (join) on exactly one white ball (in the case of overlap of two runs of black balls), (2) overlap (join) on exactly one black ball (in the case of overlap of two runs of white balls), or (3) overlap (join) on a pair of two adjacent balls of different colors (in the case of overlap of one run of white and one run of black balls).

Definition 5.2: For any vector $(w_1, w_2, \dots, w_{r+s})$ which specifies a compatible arrangement of $\sum_{\alpha=1}^{r+s} n(w_{\alpha})$ runs, associate the vector (ρ_1, ρ_2, ρ_3) where ρ_1 equals the number of times overlap (join) of type 1, defined above, occurs among these runs, $i = 1, 2, 3$.

Note that $0 \leq \rho_1 \leq \sum_{\alpha=r+1}^{r+s} n(w_{\alpha}) - 1$, $0 \leq \rho_2 \leq \sum_{\alpha=1}^r n(w_{\alpha}) - 1$,

$0 \leq \rho_3 \leq \min \left[\sum_{\alpha=1}^r n(w_{\alpha}) - 1, \sum_{\alpha=r+1}^{r+s} n(w_{\alpha}) - 1 \right]$ and that, as

well, $0 \leq \rho_1 + \rho_3 \leq \sum_{\alpha=r+1}^{r+s} n(w_\alpha) - 1$, $0 \leq \rho_2 + \rho_3 \leq \sum_{\alpha=1}^r n(w_\alpha) - 1$.

Definition 5.3: Let (ρ_1, ρ_2, ρ_3) be given. Define $N(\rho_1, \rho_2, \rho_3)$ to be the number of distinct vectors $(w_1, w_2, \dots, w_{r+s})$ which can be formed such that the associated $n(w_\alpha)$ runs, $n(w_\alpha)$ of type α which each w_α specifies, are compatible and have ρ_i joins of type i , $i = 1, 2, 3$.

Definition 5.4: Two starting positions on the circle are defined to be close if the set of positions occupied by the run starting at the first position overlaps at least in part the set of positions on the circle occupied by the run starting at the second position.

Definition 5.5: An equivalence class is characterized by a relation R which is reflexive, symmetric and transitive. Note that closeness is an equivalence relation so, by a theorem of Birkhoff and MacLane (4, p. 161), all n positions on the circle are divided by the relation closeness into mutually exclusive equivalence classes which will be called close classes.

C. Lemmas

Several lemmas which will be used in proving asymptotic multivariate independent Poissonness for runs are now given.

Lemma 5.1: Suppose $\lim_{n \rightarrow \infty} E\{I[k_1(n), n]\} = C_1 > 0$. Then

$k_1(n)$ is of order at most $n^{1/2} \ln(n)$.

Proof:

1. Let $\{n_l\}$ denote subsequence of $\{n\}$ on which $k_1(n)$ is 1 or 2.
2. Let $\{n_k\}$ denote subsequence of $\{n\}$ on which $k_1(n) \geq 2$.
2. It is clear that on $\{n_l\}$, $k_1(n)$ is of order $< n^{1/2} \ln(n)$.

3. For the other, non-trivial, subsequence $\{n_k\}$, the argument is as follows, with n_k replaced by n .

4. By definition and hypothesis, $E\{I[k_1(n), n]\} = n[p(n)]^{k_1(n)} [q(n)]^2 = C_1 \phi(n)$ where $\phi(n) > 0$ and $\rightarrow 1$ as $n \rightarrow \infty$.

5. Taking logarithms $k_1(n) = \frac{\ln n + 2 \ln q(n) - \ln C_1 \phi(n)}{-\ln p(n)}$.

6. Since $0 \leq p(n)$, $q(n) \leq 1$, from step 4 $n[p(n)]^2 \geq C_1 \phi(n)$ so $p(n) \geq \left[1 - \frac{C_1 \phi(n)}{n}\right]^{1/2}$.

7. Therefore, taking logarithms, $\ln p(n) \geq -\left[\frac{C_1 \phi(n)}{n}\right]^{\frac{1}{2}}$

except for terms of order less than the order of $\left[\frac{\phi(n)}{n}\right]^{1/2}$, since $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$ for $-1 \leq x < 1$.

8. Therefore $\frac{1}{-\ln p(n)} \leq \left[\frac{C_1 \phi(n)}{n}\right]^{-1/2}$.

9. Substitution into step 6 gives $k_1(n) \leq \lceil \ln n +$

$$2 \ln q(n) - \ln c_1 \phi(n) \leq \left[\frac{c_1 \phi(n)}{n} \right]^{-1/2} \leq \frac{n^{1/2} \ln n}{[c_1 \phi(n)]^{1/2}}$$

$$= o(n^{1/2} \ln n).$$

10. Hence, since the lemma is true for every subsequence of $\{n\}$ it is true for $\{n\}$ (28, p. 80).

Q.E.D.

Note that if $\lim_{n \rightarrow \infty} E\{J[\ell_j(n), n]\} = D_j > 0$, then $\ell_j(n)$ is of order at most $n^{1/2} \ln(n)$ by a similar argument.

Lemma 5.2: Suppose $\lim_{n \rightarrow \infty} E\{I[k_1(n), n]\} = c_1 > 0$ and

$\lim_{n \rightarrow \infty} E\{J[\ell_j(n), n]\} = D_j > 0$ for all i and j . Then (a) if

$r \geq 1$, $\lim_{n \rightarrow \infty} nq(n) \rightarrow \infty$; (b) if $s \geq 1$, $\lim_{n \rightarrow \infty} np(n) \rightarrow \infty$;

(c) if $r, s \geq 1$, $\lim_{n \rightarrow \infty} np(n)q(n) \rightarrow \infty$.

Proof (a):

1. By hypothesis and definition $E\{I[k_1(n), n]\} = n[p(n)]^{k_1(n)} [q(n)]^2 = c_1 \phi(n)$ where $\phi(n) > 0$ and $\rightarrow 1$ as $n \rightarrow \infty$.

2. Since $0 \leq p(n) \leq 1$, $n[q(n)]^2 \geq n[p(n)]^{k_1(n)}$.
 $[q(n)]^2 = c_1 \phi(n)$.

3. Therefore $nq(n) \geq [n c_1 \phi(n)]^{1/2}$ so that $\lim_{n \rightarrow \infty} nq(n) \geq c_1^{1/2} \lim_{n \rightarrow \infty} n^{1/2} \rightarrow \infty$.

Proof (b):

Similar to proof (a).

Proof (c):

1. By hypothesis and definition $E\{I[k_1(n), n]\} = n[p(n)]^{k_1(n)} [q(n)]^2 = C_1 \phi(n)$ and $E\{J[l_j(n), n]\} = n[p(n)]^2 [q(n)]^{l_j(n)} = D_j \psi(n)$ where $\phi(n), \psi(n) > 0$ and $\phi(n), \psi(n) \rightarrow 1$ as $n \rightarrow \infty$.

2. Since $k_1(n), l_j(n) \geq 1$ and $0 \leq p(n), q(n) \leq 1$, $n p(n) [q(n)]^2 \geq C_1 \phi(n)$, $n [p(n)]^2 q(n) \geq D_j \psi(n)$.

3. Multiplying, $[n p(n) q(n)]^3 \geq [C_1 D_j \phi(n) \psi(n)] \cdot n$ so $\lim_{n \rightarrow \infty} n p(n) q(n) \geq [C_1 D_j]^{1/3} \lim_{n \rightarrow \infty} n^{1/3} = \infty$.

Q.E.D.

Lemma 5.3: Let (ρ_1, ρ_2, ρ_3) be given. Then $N(\rho_1, \rho_2, \rho_3) \leq$

$$O\left(n^{\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i}\right).$$

Proof:

1. (ρ_1, ρ_2, ρ_3) is given. Interpretation of Definition 5.3 shows $N(\rho_1, \rho_2, \rho_3)$ is the number of ways $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ starting positions on the circle can be chosen, $n(w_\alpha)$ distinct among themselves, such that for each choice the corresponding $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ runs are compatible and have (ρ_1, ρ_2, ρ_3) joins.

2. $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ starting positions whose associated runs have (ρ_1, ρ_2, ρ_3) joins form $\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i$ close classes,

since (a) the minimum number of close classes whose associated

runs have (ρ_1, ρ_2, ρ_3) joins is 1 if ρ_3 is odd, and consists of $(\rho_1 + 1) + (\rho_2 + 1) + (\rho_3 - 1)$ starting positions, or is 2 if ρ_3 is even, and consists of $(\rho_1 + 1) + (\rho_2 + 1) + \rho_3$ starting positions; (b) the maximum number of close classes whose associated runs have (ρ_1, ρ_2, ρ_3) joins is $\sum_{i=1}^3 \rho_i$ and involves $2 \sum_{i=1}^3 \rho_i$ starting positions; (c) in all cases the total number of close classes whose associated runs have (ρ_1, ρ_2, ρ_3) joins is $\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i$.

3. Then $N(\rho_1, \rho_2, \rho_3)$ can be considered as enumerating all possible distinct arrangements of the $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ starting positions, $n(w_\alpha)$ for runs of type α , which form $\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i$ close classes -- a finite number depending on $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ and (ρ_1, ρ_2, ρ_3) , call it $K \left[\sum_{\alpha=1}^{r+s} n(w_\alpha), \rho_1, \rho_2, \rho_3 \right]$ -- and, for each of these, counting the number of ways the $\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i$ close classes can be arranged on the circle.

4. A set of c close classes can be arranged in the circle in $\leq \binom{n}{c} = O(n^c)$ ways.

5. Hence,

$$N(\rho_1, \rho_2, \rho_3) \leq K \left[\sum_{\alpha=1}^{r+s} n(w_\alpha), \rho_1, \rho_2, \rho_3 \right] O(n^{\sum_{\alpha=1}^{r+s} n(w_\alpha) - \sum_{i=1}^3 \rho_i}).$$

Q.E.D.

$$\text{Lemma 5.4: } N(0,0,0) = \frac{n^{\sum_{\alpha=1}^{r+s} n(w_\alpha)}}{\prod_{\alpha=1}^{r+s} n(w_\alpha)!} + o(n^{\sum_{\alpha=1}^{r+s} n(w_\alpha)}).$$

Proof:

1. $K \left[\sum_{\alpha=1}^{r+s} n(w_\alpha), 0, 0, 0 \right] = 1$, since there is only one way to choose $\sum_{\alpha=1}^{r+s} n(w_\alpha)$ starting positions in order to have no joins of any sort, namely, all non-overlapping.

$$2. N(0,0,0) \leq \prod_{\alpha=1}^{r+s} \binom{n - n(w_{\alpha-1})}{n(w_\alpha)} = n!.$$

$$\frac{1}{\prod_{\alpha=1}^{r+s} [n(w_\alpha)! (n - \sum_{\alpha=1}^{r+s} n(w_\alpha))!]},$$

where $n(w_0) = 0$, since the right hand side assumes all runs occupy only 1 position.

$$3. N(0,0,0) \geq \frac{n(n-L)(n-2L) \cdots (n - \sum_{\alpha=1}^{r+s} n(w_\alpha)L + L)}{\prod_{\alpha=1}^{r+s} n(w_\alpha)!}$$

where $L/2 = \max_{\alpha} [k_{\alpha}(n), \ell_{\alpha}(n)] + 1 \leq O(n^{1/2} \ln n)$ by Lemma 5.1.

4. Since both lower and upper bounds for $N(0,0,0)$ have desired form, the conclusion follows.

Q.E.D.

D. Theorem on "Runs"

Using the above lemmas the following theorem is proved.

Theorem 5.1: Consider the $r+s$ -dimensional random variable $I(n)$ in Definition 5.1. Then, if $\lim_{n \rightarrow \infty} E[I(n)] = C =$

$(C_1, C_2, \dots, C_{r+s})$ where $C_\alpha > 0$, $I(n)$ converges in distribution to the joint distribution of $r+s$ mutually independent Poisson variables, with parameters respectively C_α .

Proof:

1. Consider the $v = (v_1, v_2, \dots, v_{r+s})$ factorial moment of $I(n)$, say $\mu_{(n)}(v)$.

2. By Theorem 4.1 $\mu_{(n)}(v) = [v!] S(v)$ where $S(v) = \sum_{V(v)} P(w_1, w_2, \dots, w_{r+s})$, $V(v)$ and $P(w_1, w_2, \dots, w_{r+s})$ have been interpreted above.

$$3. S_{(n)}(v) = \sum_{\text{all } (\rho_1, \rho_2, \rho_3)} N(\rho_1, \rho_2, \rho_3).$$

$$\frac{\prod_{\alpha=1}^r \left\{ [p(n)]^{k_\alpha(n)} [q(n)] \right\}^{v_\alpha}}{[p(n)]^{\rho_1 + \rho_3} [q(n)]^{\rho_2 + \rho_3}} \cdot \prod_{\alpha=r+1}^{r+s} \left\{ [p(n)]^2 [q(n)]^{l_\alpha(n)} \right\}^{v_\alpha}$$

where (ρ_1, ρ_2, ρ_3) , $N(\rho_1, \rho_2, \rho_3)$ are defined in Definitions 5.2

and 5.3 respectively.

4. By hypothesis and definition

$$E[I_{\alpha}(n)] = c_{\alpha} \phi_{\alpha}(n) = \begin{cases} n[p(n)]^{k_{\alpha}(n)} [q(n)]^2 & \alpha = 1, 2, \dots, r, \\ n[p(n)]^2 [q(n)]^{\ell_{\alpha}(n)} & \alpha = r+1, r+2, \dots, r+s \end{cases}$$

where $\phi_{\alpha}(n) > 0$ and $\rightarrow 1$ as $n \rightarrow \infty$. Therefore write

$$S(v) = \prod_{\alpha=1}^{r+s} [c_{\alpha} \phi_{\alpha}(n)]^{v_{\alpha}} \sum_{\substack{\text{all} \\ (\rho_1, \rho_2, \rho_3)}} \frac{N(\rho_1, \rho_2, \rho_3)}{n^{\sum_{\alpha=1}^{r+s} v_{\alpha}} [p(n)]^{\rho_1 + \rho_3} [q(n)]^{\rho_2 + \rho_3}}.$$

5. Replace $N(\rho_1, \rho_2, \rho_3)$ by the expressions given in Lemmas 5.3 and 5.4. Then

$$\prod_{\alpha=1}^{r+s} [c_{\alpha} \phi_{\alpha}(n)]^{v_{\alpha}} \left[\frac{1}{\prod_{\alpha=1}^{r+s} v_{\alpha}!} + \frac{o(n)}{\sum_{\alpha=1}^{r+s} v_{\alpha}} \right] \leq S(v) \leq$$

$$\prod_{\alpha=1}^{r+s} [c_{\alpha} \phi_{\alpha}(n)]^{v_{\alpha}} \left[\frac{1}{\prod_{\alpha=1}^{r+s} v_{\alpha}!} + \frac{o(n)}{\sum_{\alpha=1}^{r+s} v_{\alpha}} \right] +$$

$$\sum_{\substack{(\rho_1, \rho_2, \rho_3) \\ \neq (0,0,0)}} \left[\frac{K \left(\sum_{\alpha=1}^{r+s} v_{\alpha}, \rho_1, \rho_2, \rho_3 \right) \sum_{\alpha=1}^{r+s} v_{\alpha}}{[np(n)]^{\rho_1} [nq(n)]^{\rho_2} [np(n)q(n)]^{\rho_3} \sum_{\alpha=1}^{r+s} v_{\alpha}} \right]$$

6. When $(\rho_1, \rho_2, \rho_3) \neq (0,0,0)$, by Lemma 5.2, then at least one term with corresponding exponent $\rho > 0$ -- $np(n)$, $nq(n)$, or $np(n)q(n)$ -- approaches ∞ . Since (ρ_1, ρ_2, ρ_3) is bounded, one has

$$\lim_{n \rightarrow \infty} S(v) = \prod_{\alpha=1}^{r+s} \frac{C_{\alpha}^{v_{\alpha}}}{v_{\alpha}!}.$$

7. From step 2, then, $\lim_{n \rightarrow \infty} \mu_{(n)}(v) = \prod_{\alpha=1}^{r+s} C_{\alpha}^{v_{\alpha}}$, the v -th factorial moment of the joint distribution of $r+s$ independent Poisson variables, the univariate marginal having parameter C_{α} .

8. Since $\mu_{(n)}(v)$ exist for all v and are bounded independently of n (as can be seen from consideration of the expression in step 5), the corresponding moments about the origin also exist and are bounded independently of n .

9. Therefore steps 7 and 8 and Theorem 3.4 show that the random variable $I(n)$ converges in distribution to that of $r+s$ independent Poissons, with parameters respectively C_{α} .

Q.E.D.

E. Corollaries

The asymptotic distributions of various combinations of these runs are stated in the following corollaries, where $I(n)$ is defined as in Definition 5.1.

Corollary 5.1: If $\lim_{n \rightarrow \infty} E[I(n)] = C > 0$, then the distribution of $\sum_{\alpha=1}^{r+s} I_{\alpha}(n)$ converges to the univariate Poisson with parameter $\sum_{\alpha=1}^{r+s} C_{\alpha}$.

Proof:

1. Theorem 5.1 and Corollary 2.1.

Q.E.D.

This corollary states that the asymptotic distribution of the sum of numbers of runs of arbitrary lengths is Poisson.

In particular, when $r = s = 1$ and $k_1(n) = \ell_1(n)$, we have essentially the result of von Mises (22), namely, that the asymptotic distribution of number of runs of the same length is Poisson, where the number includes both white and black ball run counts.

Corollary 5.2: If $\lim_{n \rightarrow \infty} E[I(n)] = C > 0$, the bivariate distribution of number of black ball runs of length $k(n)$ and number of runs of length $k(n)$ is asymptotically semi-Poisson with parameters $a_2 = C_2$, $a_{12} = C_1$.

Proof:

1. Let $I_1(n)$, $I_2(n)$ be number of runs of length $k(n)$ of black and white balls, respectively.

2. By Theorem 5.1 the joint distribution of $[I_1(n), I_2(n)]$ is asymptotically bivariate independent Poisson with parameters C_1, C_2 .

3. By Corollary 2.2 the joint distribution of $[I_1(n), I_1(n) + I_2(n)]$ is asymptotically bivariate Poisson. It has means $C_1, C_1 + C_2$, and covariance C_1 by appeal to its characteristic function.

4. Analogy between this notation and that of the c.f. in Equation (1) shows that $a_1 = 0$, so this particular bivariate Poisson distribution is the semi-Poisson, with parameters $a_2 = C_2, a_{12} = C_1$.

Q.E.D.

Corollary 5.3: If $\lim_{n \rightarrow \infty} E[I(n)] = C > 0$, the asymptotic

multivariate distribution of numbers of sums of runs of arbitrary type and length is multivariate Poisson.

Proof:

1. Theorem 5.1 and Corollary 2.2.

Q.E.D.

VI. POISSON LIMITS OF CONFIGURATION DISTRIBUTIONS

A. Introduction

It has been shown previously that the asymptotic distribution of number of "runs" in the usual sense under the restriction of constant expectations in the limit is multivariate independent Poisson. The concept of "runs" can be generalized, however, to that of configurations where a configuration is specified by its pattern as well as by the number of white and black balls it contains. In contrast to numbers of runs, numbers of configurations need not be asymptotically independent random variables. It will be shown that under certain restrictions the asymptotic joint distribution of the number of two very simple configurations is bivariate correlated Poisson.

B. Definitions

Definition 6.1: Let $M_\alpha(n)$ equal the number of configurations $C_\alpha(n)$, $\alpha = 1, 2$, which materialize on the circle, where $C_\alpha(n)$ is a succession of $k_\alpha(n) \geq 1$ white balls immediately followed by a succession of $\ell_\alpha(n) \geq 1$ black balls, $[k_1(n), \ell_1(n)] \neq [k_2(n), \ell_2(n)]$.

Definition 6.2: Let $\overline{k(n)} = \max_\alpha [k_\alpha(n)]$, $\underline{k(n)} = \min_\alpha [k_\alpha(n)]$, $\overline{\ell(n)} = \max_\alpha [\ell_\alpha(n)]$, $\underline{\ell(n)} = \min_\alpha [\ell_\alpha(n)]$.

Recall the definitions of Section IV. B. Interpret them in terms of configurations as was done in Section V. B for "runs". Note for configurations $C_\alpha(n)$ considered here, non-zero probability $P(w_1, w_2)$ occurs either when (w_1, w_2) specifies an arrangement of $\sum_{\alpha=1}^2 n(w_\alpha)$ configurations with no overlap or when (w_1, w_2) specifies an arrangement of $\sum_{\alpha=1}^2 n(w_\alpha)$ configurations such that pairs of configurations, $C_1(n)$, $C_2(n)$, overlap on $k(n)$ white balls, $l(n)$ black balls.

Definition 6.3: For any vector (w_1, w_2) which specifies a compatible arrangement of $\sum_{\alpha=1}^2 n(w_\alpha)$ configurations, associate the value ρ which equals the number of times overlap of pairs of configurations $C_1(n)$, $C_2(n)$ occurs among these configurations, $0 \leq \rho \leq \min_{\alpha} [n(w_\alpha)]$.

Definition 6.4: Let ρ be given. Define $N(\rho)$ to be the number of distinct vectors (w_1, w_2) such that the configurations associated with the $\sum_{\alpha=1}^2 n(w_\alpha)$ starting positions which each (w_1, w_2) specifies are compatible and have ρ joins or overlaps.

C. Lemmas

The following lemmas will be used in proving the asymptotic joint distribution of $\mathbf{M}(n) = [\mathbf{M}_1(n), \mathbf{M}_2(n)]$ is bivariate Poisson.

Lemma 6.1: If $\lim_{n \rightarrow \infty} E[\mathbf{M}_\alpha(n)] = K_\alpha > 0$, $\alpha = 1, 2$, then $0 < p(n) < 1$.

Proof:

1. By Definition 6.1, $(k_\alpha(n), l_\alpha(n)) \geq (1, 1)$, so
 $0 < K_\alpha \phi_\alpha(n) = E\{\mathbf{M}_\alpha(n)\} = n [p(n)]^{k_\alpha(n)} [q(n)]^{l_\alpha(n)}$
 $\leq n p(n) q(n)$ for $\alpha = 1, 2$, and where $\phi_\alpha(n) > 0$, $\rightarrow 1$ as $n \rightarrow \infty$.

2. Then, if $p(n) = 0$, R.H.S. = 0, a contradiction; if $p(n) = 1$, then $q(n) = 0$, and R.H.S. again = 0, a contradiction.
 Q.E.D.

Lemma 6.2: If $\lim_{n \rightarrow \infty} E[\mathbf{M}_\alpha(n)] = K_\alpha > 0$, $\alpha = 1, 2$, and if

at least one configuration contains more than 1 white ball and at least one configuration contains more than 1 black ball, then $k_\alpha(n) \leq O(n^{1/2} \ln n)$ and $l_\alpha(n) \leq O(n^{1/2} \ln n)$.

Proof:

1. Suppose $\mathbf{M}_1(n)$ contains $k_1(n) \geq 2$ white balls and $\mathbf{M}_2(n)$ contains $l_2(n) \geq 2$ black balls.

2. Then

$$K_1 \phi_1(n) = n [p(n)]^{k_1(n)} [q(n)]^{l_1(n)} \leq n [p(n)]^2$$

and

$$K_2 \phi_2(n) = n [p(n)]^{k_2(n)} [q(n)]^{\ell_2(n)} \leq n [q(n)]^2$$

so that

$$\left[\frac{K_1 \phi_1(n)}{n} \right]^{1/2} \leq p(n) \leq 1 - \left[\frac{K_2 \phi_2(n)}{n} \right]^{1/2}$$

and

$$\left[\frac{K_2 \phi_2(n)}{n} \right]^{1/2} \leq q(n) \leq 1 - \left[\frac{K_1 \phi_1(n)}{n} \right]^{1/2}$$

where $\phi_1(n), \phi_2(n) > 0$ and $\rightarrow 1$ as $n \rightarrow \infty$.

3. For any configuration then such that $K\phi(n) = n [p(n)]^{k(n)} [q(n)]^{\ell(n)}$ one has $\frac{K\phi(n)}{n} \leq [p(n)]^{k(n)}$ and $\frac{K\phi(n)}{n} \leq [q(n)]^{\ell(n)}$ so that, by step 2 above,

$$\begin{aligned} k(n) &\leq \frac{\ln \left[\frac{n}{K\phi(n)} \right]}{-\ln p(n)} \leq \frac{\ln \left[\frac{n}{K\phi(n)} \right]}{-\ln \left(1 - \left[\frac{K_2 \phi_2(n)}{n} \right]^{1/2} \right)} = \frac{\ln \left[\frac{n}{K\phi(n)} \right]}{\left[\frac{K_2 \phi_2(n)}{n} \right]^{1/2}} \\ &= O(n^{1/2} \ln n) \end{aligned}$$

and similarly $\ell(n) \leq O(n^{1/2} \ln n)$.

Q.E.D.

Lemma 6.3: If $\lim_{n \rightarrow \infty} E[M_\alpha(n)] = K_\alpha > 0$, $\alpha = 1, 2$, and if $\lim_{n \rightarrow \infty} \frac{\overline{k(n)} - k(n)}{[p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)} - \ell(n)}} = c$, $0 \leq c \leq 1$, then $\lim_{n \rightarrow \infty} n [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} = [K_1 K_2 c]^{1/2} \geq 0$.

Proof:

1. Consider a subsequence of $\{n\}$ such that $[\overline{k(n)}, \overline{\ell(n)}] = [k_1(n), \ell_2(n)]$.

2. For this subsequence $n [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} = K_1 \phi_1(n) [q(n)]^{\overline{\ell(n)} - \underline{\ell(n)}} = K_2 \phi_2(n) [p(n)]^{\overline{k(n)} - \underline{k(n)}}$.

3. Then $[p(n)]^{\overline{k(n)} - \underline{k(n)}} [q(n)]^{\underline{\ell(n)} - \overline{\ell(n)}} = \frac{K_1 \phi_1(n)}{K_2 \phi_2(n)}$,

where division is possible by Lemma 6.1.

4. However, by hypothesis, $[p(n)]^{\overline{k(n)} - \underline{k(n)}} [q(n)]^{\underline{\ell(n)} - \overline{\ell(n)}} = C \psi(n)$.

5. This and step 3 give $[q(n)]^{\overline{\ell(n)} - \underline{\ell(n)}} = \left[\frac{K_2 C \phi_2(n) \psi(n)}{K_1 \phi_1(n)} \right]^{1/2}$.

6. From step 2, then $\lim_{n \rightarrow \infty} n [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} = [K_1 K_2 C]^{1/2}$, which was to be proved. A similar demonstration holds for subsequence of $\{n\}$ where $[\overline{k(n)}, \overline{\ell(n)}] = [k_2(n), \ell_1(n)]$.

7. Consider a subsequence of $\{n\}$ such that $[\overline{k(n)}, \overline{\ell(n)}] = [k_1(n), \ell_1(n)]$.

8. For this subsequence, $n \left[p(n) \right]^{\overline{k(n)}} \left[q(n) \right]^{\overline{\ell(n)}} = K_1 \phi_1(n)$.

9. Since $\left[p(n) \right]^{\overline{k(n)} - \underline{k(n)}} \left[q(n) \right]^{\overline{\ell(n)} - \underline{\ell(n)}} = \frac{K_1 \phi_1(n)}{K_2 \phi_2(n)}$

$= C \cdot \dagger(n)$ by hypothesis, for this subsequence $\left[K_1 K_2 C \right]^{1/2} = K_1$, the result in step 8, so that the lemma is true for this subsequence also. A similar demonstration holds for the subsequence of $\{n\}$ such that $\left[\overline{k(n)}, \overline{\ell(n)} \right] = \left[k_2(n), \ell_2(n) \right]$.

Q.E.D.

Lemma 6.4: Let ρ be given. Then

$$N(\rho) = \left[\prod_{\alpha=1}^2 \binom{n(w_\alpha)}{\rho} \right] \left[\frac{\sum_{\alpha=1}^2 n(w_\alpha) - \rho}{(n(w_1) - \rho)! (n(w_2) - \rho)! \rho!} + o(n) \right].$$

Proof:

1. $N(\rho)$ in Definition 6.4 can be interpreted as the number of ways $\sum_{\alpha=1}^2 n(w_\alpha)$ starting positions, $n(w_\alpha)$ for configuration type $C_\alpha(n)$, $\alpha = 1, 2$, whose associated configurations have ρ joins, can form $\sum_{\alpha=1}^2 n(w_\alpha) - \rho$ close classes on the circle, since ρ joins can be formed only by ρ pairs of different configurations.

2. ρ pairs can be formed in $\prod_{\alpha=1}^2 \binom{n(w_\alpha)}{\rho}$ distinct ways.

3. Each set of $\sum_{\alpha=1}^2 n(w_\alpha) - \rho$ close classes can be arranged on the circle in

$$W(\rho) = \frac{\sum_{\alpha=1}^2 n(w_\alpha) - \rho}{\binom{n}{n(w_1) - \rho} \binom{n}{n(w_2) - \rho} \rho!}$$

ways, since:

(a) all classes are of length > 1 and three distinct types of classes are selected, namely $n(w_1) - \rho$ of type $C_1(n)$, $n(w_2) - \rho$ of type $C_2(n)$, ρ a combination of $C_1(n)$, $C_2(n)$, so that

$$W(\rho) \leq \binom{n}{n(w_1) - \rho} \binom{n - n(w_1) + \rho}{n(w_2) - \rho} \binom{n - n(w_1) - n(w_2) + 2\rho}{\rho};$$

(b) all classes are of length $< L = \overline{k(n)} + \overline{\ell(n)} + 1 \leq O[n^{1/2} \ln n]$ by Lemma 6.2, so that

$$W(\rho) \geq \frac{\sum_{\alpha=1}^2 n(w_\alpha) - \rho - 1}{\binom{n}{n(w_1) - \rho} \binom{n}{n(w_2) - \rho} \rho!} \cdot \frac{n(n - 2L)(n - 2^2L) \cdots (n - 2^L L)}{L}.$$

4. By steps 1, 2 and 3 the lemma follows.

Q.E.D.

D. Theorem on Configurations

The above lemmas are used to prove the following theorem.

Theorem 6.1: Consider the vector random variable $M(n) = [M_1(n), M_2(n)]$, $M_\alpha(n)$ defined in Definition 6.1, $\alpha = 1, 2$. Assume $[k_1(n), k_2(n)] \neq [1, 1]$ and $[\ell_1(n), \ell_2(n)] \neq [1, 1]$. If $\lim_{n \rightarrow \infty} E[M_\alpha(n)] = K_\alpha > 0$, and if

$$\lim_{n \rightarrow \infty} [p(n)]^{\overline{k(n)} - k(n)} [q(n)]^{\overline{\ell(n)} - \ell(n)} = c, \quad 0 \leq c \leq 1, \text{ then}$$

$M(n)$ converges in distribution to the bivariate Poisson with correlation coefficient $c^{1/2}$.

Proof:

1. Consider the $v = (v_1, v_2)$ -th factorial moment of $M(n)$, say $\mu_{(n)}(v)$.

2. By Theorem 4.1, $\mu_{(n)}(v) = [v!] S(v)$.

3. By Definitions 4.5, 6.2, 6.3, 6.4 one has

$$\begin{aligned} S(v) &= \sum_{V(v)} P(w_1, w_2) \\ &= \sum_{\rho=0}^{\underline{v}=\min(v_\alpha)} N_\rho \prod_{\alpha=1}^2 \left\{ [p(n)]^{k_\alpha(n)} [q(n)]^{\ell_\alpha(n)} \right\}^{v_\alpha - \rho} \\ &\quad \left\{ [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} \right\}^\rho. \end{aligned}$$

4. By Lemma 6.4 and assumption of constant expectations, where $\phi_\alpha(n) > 0, \rightarrow 1$ as $n \rightarrow \infty$, $\alpha = 1, 2$, write

$$S(v) = \sum_{p=0}^v \frac{1}{(v_1-p)! (v_2-p)! p!} \left\{ \frac{\sum_{\alpha=1}^2 v_{\alpha}^{-p}}{v_1! v_2!} + o(n \sum_{\alpha=1}^2 v_{\alpha}^{-p}) \right\} \\ \left\{ \frac{\prod_{\alpha=1}^2 [K_{\alpha} \theta_{\alpha}(n)]^{v_{\alpha}-p}}{\sum_{\alpha=1}^2 v_{\alpha}^{-2p}} \right\} \left\{ [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} \right\}^p$$

$$= \sum_{p=0}^v \frac{1}{(v_1-p)! (v_2-p)! p!} \left\{ \prod_{\alpha=1}^2 \frac{[K_{\alpha} \theta_{\alpha}(n)]^{v_{\alpha}-p}}{v_1! v_2!} \right. \\ \left. + \frac{o(n \sum_{\alpha=1}^2 v_{\alpha}^{-2p})}{\sum_{\alpha=1}^2 v_{\alpha}^{-2p}} \right\} \left\{ n [p(n)]^{\overline{k(n)}} [q(n)]^{\overline{\ell(n)}} \right\}^p.$$

5. By assumption $[p(n)]^{\overline{k(n)} - \overline{k(n)}}$ $[q(n)]^{\overline{\ell(n)} - \overline{\ell(n)}}$ = $O(\psi(n))$ where $\psi(n) > 0$, $\rightarrow 1$ as $n \rightarrow \infty$, and Lemma 6.3, one has

$$\lim_{n \rightarrow \infty} S(v) = \sum_{p=0}^v \frac{1}{(v_1-p)! (v_2-p)! p!} \left\{ \prod_{\alpha=1}^2 \frac{[K_{\alpha}]^{v_{\alpha}-p}}{v_1! v_2!} \right\} \left\{ K_1 K_2 C \right\}^{p/2} \\ = \sum_{p=0}^v \frac{K_1^{v_1-p} K_2^{v_2-p} \{ [K_1 K_2 C]^{1/2} \}^p}{(v_1-p)! (v_2-p)! p!}.$$

6. Hence, by step 2 and Campbell's (6, p. 20) expression for the factorial moments of a correlated bivariate Poisson, one sees that $\lim_{n \rightarrow \infty} \mu_{(n)}(v)$ is the factorial moment of a bivariate Poisson with means K_1, K_2 , covariance $[K_1 K_2 C]^{1/2}$, and correlation coefficient $r = \left[\frac{K_1 K_2 C}{K_1 K_2} \right]^{1/2} = C^{1/2}$.

7. Since $\mu_{(n)}(v)$ exist for all v and are bounded independently of n (as can be seen from consideration of the expression in step 4) the corresponding moments about the origin also exist and are bounded independently of n .

8. Therefore, from steps 6, 7 and Theorem 3.4, it is seen that the random variable $M(n)$ converges in distribution to a bivariate Poisson with correlation coefficient $C^{1/2}$.
Q.E.D.

E. Corollaries

It is interesting to note the several special cases of this simple type of configuration which lead to independent, correlated or equivalent random variables. These cases will be presented in the following corollaries. Let $M(n) = [M_1(n), M_2(n)]$ be defined as in Definition 6.1. Assume the conditions of Theorem 6.1 hold.

Corollary 6.1: When $C = 0$, $0 < C < 1$, or $C = 1$, $M(n)$ is respectively asymptotically bivariate independent Poisson, bivariate correlated Poisson, or bivariate equivalent Poisson.

Proof:

1. Definition of $C^{1/2}$, Theorem 6.1.

Q.E.D.

Corollary 6.2: When $K_1 < K_2$ and $C = K_1/K_2$ the asymptotic distribution of $M(n)$ is semi-Poisson with parameters $a_2 = K_2 - K_1$ and $a_{12} = K_1$.

Proof:

1. By definition of $C^{1/2}$, the correlation coefficient

$$r = C^{1/2} = \left[\frac{K_1}{K_2} \right]^{1/2}.$$

2. Analogy to the notation of Equation (1) gives

$$a_1 + a_{12} = K_1$$

$$a_2 + a_{12} = K_2$$

$$a_{12} = K_1$$

so that $a_1 = 0$, and the asymptotic distribution of $M(n)$ is semi-Poisson with parameters $a_2 = K_2 - K_1$ and $a_{12} = K_1$.

Q.E.D.

Corollary 6.3: When $C = 0$, $M_1(n) + M_2(n)$ is asymptotically univariate Poisson with parameter $K_1 + K_2$; when $0 < C \leq 1$, $M_1(n) + M_2(n)$ is asymptotically univariate multiple Poisson with parameters $K_1 + K_2 - 2[K_1 K_2 C]^{1/2}$ and $[K_1 K_2 C]^{1/2}$.

Proof:

1. By Theorem 6.1 the c.f. of $[M_1(n), M_2(n)]$ is asymptotically bivariate Poisson as in Equation (1) with $a_1 + a_{12}$

$$= K_1, a_2 + a_{12} = K_2, a_{12} = [K_1 K_2 C]^{1/2}.$$

2. The distribution of the sum $M_1(n) + M_2(n)$ has c.f. $c(t, t)$ with parameters $a_1 + a_2 = K_1 + K_2 - 2[K_1 K_2 C]^{1/2}$ and $a_{12} = [K_1 K_2 C]^{1/2}$.

3. By Corollary 2.3, $M_1(n) + M_2(n)$ is asymptotically multiple Poisson.

Q.E.D.

Corollary 6.4: When $C = 1$ and $K_1 = K_2 = K$, $M_1(n) + M_2(n)$ is asymptotically multiple Poisson with parameter K^2 and non-zero probability only on the non-negative even integers.

Proof:

1. By Corollary 6.3 in this case $M_1(n) + M_2(n)$ has asymptotic distribution with c.f. $\exp \{K^2(z^2 - 1)\}$ where $z = \exp \{it\}$. This is the c.f. of a univariate Poisson distribution which counts doublets (12, p. 271).

Q.E.D.

VII. LIMIT DISTRIBUTIONS OF ARBITRARY CONFIGURATIONS WITH FIXED LENGTHS

A. Introduction

This section discusses briefly the asymptotic multivariate distribution of numbers of configurations on the circle, under the conditions (1) that the number of white and black balls in each configuration are fixed, and (2) that the expected number of the configuration whose pattern has the fewest white balls is constant in the limit. This asymptotic distribution is a special case of the multivariate Poisson in the sense that it can only involve independent, equivalent, or degenerate random variables (random variables which are zero with probability one).

B. Notation

Consider white and black balls from a binomial population arranged on a circle with n positions. Let p be the probability of a white ball and omit the index n for simplicity.

Let $N(ij)$ be the number of configurations of distinct type (ij) on a circle with n positions, where $i = 1, 2, \dots, m$ is an index of the pattern of white balls (e.g. oxo and $oxoxx$ have the same i); $j = 1, 2, \dots, r_i$ is the index of the different number of black balls superimposed on the i -th pattern of white balls. Let k_i be the number of white balls in configu-

ration (ij) . Let $\underline{k} = \min_i k_i$. Let ℓ_{ij} be the number of black balls in configuration (ij) .

C. Procedure

To find the distribution of $(N_{ij}: i = 1, 2, \dots, m; j = 1, 2, \dots, r_i)$ in the limit under the conditions that all k_i , ℓ_{ij} are fixed and $\lim_{n \rightarrow \infty} np^{\underline{k}} = \lambda$, obtain expressions for its moments in the limit, show they are those of a particular multivariate Poisson distribution, and apply Theorem 3.4 to get convergence in distribution.

The details of this process will be omitted. It essentially involves showing that (1) for independence the contribution of the counts of the overlapping configurations approaches zero, (2) for equivalence the counts for the different patterns converge to the same count, and (3) for degeneracy the conditions under which the limit is taken force these particular N_{ij} to converge in probability to zero.

D. Conclusions

The following conclusions can be drawn from the expressions one obtains for the limit moments under the conditions of fixed k_i , ℓ_{ij} , and $\lim_{n \rightarrow \infty} np^{\underline{k}} = \lambda$.

Conclusion 7.1: If $k_i > \underline{k}$, then N_{ij} converges in probability to zero for all j .

Conclusion 7.2: If $k_1 = \underline{k}$, then $N_{1j} \xrightarrow{d}$ a univariate Poisson with mean λ .

Conclusion 7.3: If $k_1 = \underline{k}$, then $(N_{1j}: j = 1, 2, \dots, r_1) \xrightarrow{d}$ distribution of r_1 equivalent Poisson random variables, each with the same mean λ .

Intuitively, this can be seen as follows. For each of these N_{1j} the white ball pattern is the same (e.g. oxo, oxox, xoxox). Since the condition under which the limit is taken forces $p(n) \rightarrow 0$, the probability of a black ball $\rightarrow 1$; thus, the rare white balls are surrounded by black balls and all counts of configurations with the same pattern of white balls are counting the same thing, regardless of the number of black balls.

Conclusion 7.4: If $k_1 = \underline{k}$, then $(N_{1j}: i = 1, 2, \dots, m; j = 1, 2, \dots, r_1) \xrightarrow{d}$ distribution of m mutually independent sets, each containing respectively r_1 equivalent univariate Poisson random variables with means λ .

Conclusion 7.5: Since $(N_{1j}: j = 1, 2, \dots, r_1) \xrightarrow{d} r_1$ equivalent univariate Poissons with means λ , then $\sum_{j=1}^{r_1} N_{1j} \xrightarrow{d}$ a multiple Poisson with mean $r_1\lambda$ which has non-zero probability on the non-negative integers, nr_1 , $n = 0, 1, 2, \dots$.

VIII. THE PREVALENCE OF MULTIVARIATE POISSONS AS LIMIT
DISTRIBUTIONS OF CONFIGURATIONS

A. Introduction

The theorem presented in this section states sufficient conditions for the asymptotic joint distribution of numbers of arbitrary configurations on the circle to be multivariate Poisson, Equation (1). It is based on the fact that the only infinitely divisible distribution with Poisson marginals is the multivariate Poisson (11, p. 467).

This theorem essentially eliminates the possibility that configurations on the circle may exist whose numbers are asymptotically marginally Poisson but whose joint distribution is not multivariate Poisson.

B. Theorem

Theorem 8.1: Let $I_i(n)$, $i = 1, 2, \dots, m$ be the number of arbitrary configurations $R_i(n)$ of $k_i(n)$ white balls and $\ell_i(n)$ black balls which materialize on the circle with n positions, such that:

(1) at least one configuration contains more than 1 black ball and at least one configuration contains more than 1 white ball, so, by Lemma 6.2, $\max_i [k_i(n) + \ell_i(n)] \leq L n^{1/2} \ln n$;

(2) on any arc A of an $n + o(n)$ successive positions,

$I(a,n) = [I_1(a,n), I_2(a,n), \dots, I_m(a,n)] \xrightarrow{d}$ multivariate distribution whose marginals are Poissons with parameters $\lambda_{ia} = a\lambda_i$, $0 \leq a \leq 1$, where $I_i(a,n)$ is the number of configurations of type R_i which materialize on arc A .

Then $I(n) \xrightarrow{d}$ multivariate Poisson, where $I(n) = I(1,n)$.

Proof:

1. By assumption (2) with $a = 1$, write $I(n) \xrightarrow{d} I$, a multivariate distribution with Poisson marginals each with parameter λ_i , $i = 1, 2, \dots, m$. In order to prove this multivariate distribution is multivariate Poisson it is necessary only to show that I is infinitely divisible, since, by Dwass and Teicher (11, p. 467), the only infinitely divisible multivariate distribution with Poisson marginals is the multivariate Poisson. In other words, it is necessary only to show that for an arbitrary integer $K > 0$, the characteristic function $c(t)$ of I satisfies $c(t) = [\psi(t)]^K$ for some characteristic function $\psi(t)$.

2. Consider K given. Fix n . Divide the circle into $2K$ arcs, K of them large and K of them small, such that small arcs alternate with large ones, where the length of a large arc $\doteq \frac{n}{K} - L n^{1/2} \ln n$, length of a small arc $\doteq L n^{1/2} \ln n$, and all large arcs have exactly the same length.

3. Let $\epsilon_{1,K,n}, \epsilon_{2,K,n}, \dots, \epsilon_{K,K,n}$ count the numbers of configurations (R_1, R_2, \dots, R_m) which appear on the K small

arcs, and let $I_{1,K,n}, I_{2,K,n}, \dots, I_{K,K,n}$ count the numbers of configurations (R_1, R_2, \dots, R_m) which appear on the K large arcs. Note that $\epsilon_{i,K,n}, I_{i,K,n}$ are m -dimensional random variables.

$$4. \text{ Then } I(n) = \sum_{i=1}^K I_{i,K,n} + \sum_{i=1}^K \epsilon_{i,K,n} \text{ where all } I_{i,K,n}$$

are mutually independent by assumption (1).

$$5. \sum_{i=1}^K \epsilon_{i,K,n} \text{ converges in probability to zero. This}$$

can be shown as follows. Consider the first component of

$$\epsilon_{1,K,n}, \text{ say } \epsilon_{1,K,n}^1. \quad p \lim_{n \rightarrow \infty} \epsilon_{1,K,n}^1 = 0, \text{ because it satisfies}$$

the definition of convergence in probability. That is, for

any given $\gamma > 0$ there exists an $n_0(\gamma)$ such that

$$\Pr \{ |\epsilon_{1,K,n}^1| = 0 \} \geq 1 - \gamma \text{ for } n > n_0(\gamma).$$

(a) Choose $n_0(\gamma) = \max \{n_1[a(\gamma)], n_2(\gamma)\}$ where

(1) $a(\gamma)$ is such that for given $\gamma > 0$ it satisfies

$$\Pr \{P(a(\gamma)\lambda_1) = 0\} = \exp \{-a(\gamma)\lambda_1\} = 1 - \gamma/2, \text{ where}$$

$P(a(\gamma)\lambda_1)$ is univariate Poisson with mean $a(\gamma)\lambda_1$.

(2) $n_1[a(\gamma)]$ is such that given $a(\gamma)$,

$0 \leq \epsilon_{1,K,n}^1 \leq J_{1,a(\gamma),n}^1$ for $n > n_1[a(\gamma)]$ where $J_{1,a(\gamma),n}^1$ counts the numbers of R_1 on a large arc constructed to cover the small arc on which $\epsilon_{1,K,n}^1$ counts the numbers of R_1 .

(3) $n_2(\gamma)$ is such that given $\gamma/2 > 0$,

$$|\Pr \{0 | J_{1,a(\gamma),n}^1\} - \Pr \{0 | P(a(\gamma)\lambda_1)\}| < \gamma/2$$

for $n > n_2(\gamma)$ by assumption (2).

(b) For this $n_0(\gamma)$, $\Pr \{\epsilon_{1,K,n}^1 = 0\} \geq 1 - \gamma$ for $n > n_0(\gamma)$, because:

(1) For $n > n_1[a(\gamma)]$ by step (a-2) above, $1 \geq \Pr \{\epsilon_{1,K,n}^1 = 0\} \geq \Pr \{J_{1,a(\gamma),n}^1 = 0\}$.

(2) For $n > n_2(\gamma)$ by step (a-3) above

$$- \gamma/2 \leq \Pr \{J_{1,a(\gamma),n}^1 = 0\} - \Pr \{P(a(\gamma)\lambda_1) = 0\} \leq \gamma/2.$$

(3) Then, by steps (a-1), (b-1) and (b-2), one has $1 \geq \Pr \{\epsilon_{1,K,n}^1 = 0\} \geq 1 - \gamma$ so $\epsilon_{1,K,n}^1$ converges in probability to zero.

(c) All $\epsilon_{i,K,n}$, $i = 1, 2, \dots, K$ converge in probability to $(0, 0, \dots, 0)$ by definition of multivariate convergence in probability (8, p. 299).

Then $\sum_{i=1}^K \epsilon_{i,K,n}$ converges in probability to zero (8, p.

254), as was to be shown.

6. Because of this, the m -dimensional distribution func-

tion of $I(n) = \sum_{i=1}^K I_{i,K,n} + \sum_{i=1}^K \epsilon_{i,K,n}$ converges to the m -

dimensional limiting distribution function of $I_0(n) =$

$\sum_{i=1}^K I_{i,K,n}$ (8, p. 300). Then, by Theorem 2.2 of this thesis,

the c.f.'s of $I(n)$ and $I_0(n)$, say $c_n(t)$ and $c_{0,n}(t)$ respectively converge to the same limit c.f., say $c(t)$.

7. However, $c_{0,n}(t) = [\psi_n(t)]^K$ since all $I_{i,K,n}$ are mutually independent and identically distributed, where $\psi_n(t)$ is c.f. of $I_{i,K,n}$.

8. Therefore, $c(t) = [\psi(t)]^K$ where $\psi(t) = \lim_{n \rightarrow \infty} \psi_n(t)$; hence I is infinitely divisible.

9. Thus, by steps 1 and 8, $I_n \xrightarrow{d} \text{a multivariate Poisson}$ (11, p. 467).

Q.E.D.

IX. DERIVATION OF UNIVARIATE MULTIPLE POISSON FROM MARKOV PROCESS POSTULATES

A. Introduction

Postulates have been advanced for certain discontinuous Markov processes which yield both the univariate (12, pp. 400-402) and the multivariate (11, pp. 468-470) Poisson distributions as space distributions for fixed time. Although Dwass and Teicher (11, p. 470) allude to the possibility of an approach of the type to be presented here, it is believed that the following detailed derivation of a univariate multiple Poisson from suitable postulates for a discontinuous Markov process has not appeared before.

Consider the particular univariate multiple Poisson random variable, $X = Y + 2Z$, where Y and Z are independent univariate Poissons with means λt and ρt , respectively. The distribution of this particular univariate multiple Poisson random variable will be derived below from suitable postulates for a discontinuous Markov process.

B. Assumptions

The distribution of this particular univariate multiple Poisson will be derived as that of the number of "materializations" $X(t)$ that have occurred up to time $t \geq 0$. The following postulates are those referred to above.

- (1) At time $t = 0$, no "materializations" have occurred.
- (2) Let $h = \Delta t$ be a small interval of time. The probability of exactly 1 "materialization" in the interval $(t, t+h)$ is $\lambda h + o(h)$, $\lambda > 0$.
- (3) The probability of exactly 2 "materializations" in the interval $(t, t+h)$ is $\rho h + o(h)$, $\rho > 0$.
- (4) The probability of no "materializations" in $(t, t+h)$ is $1 - (\lambda + \rho)h + o(h)$.
- (5) The above probabilities are independent of the state of the system.
- (6) "Materializations" in non-overlapping time intervals are independent.

C. Derivation

Consider the random variable $X(t)$, $t \geq 0$. Let $P_x(t) = \Pr \{X(t) = x\}$, $x = 0, 1, 2, \dots$. In view of the above assumptions

$$P_0(t+h) = P_0(t) [1 - (\lambda + \rho)h] + o(h),$$

$$P_1(t+h) = P_1(t) [1 - (\lambda + \rho)h] + P_0(t) [\lambda h] + o(h)$$

$$P_x(t+h) = P_x(t) [1 - (\lambda + \rho)h] + P_{x-1}(t) [\lambda h] + P_{x-2}(t) [\rho h] + o(h) \text{ for } x \geq 2,$$

where

$$P_x(0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

As $h \rightarrow 0$, these lead to the following system of differential equations:

$$\frac{dP_0(t)}{dt} = -(\lambda + \rho)P_0(t),$$

$$\frac{dP_1(t)}{dt} = -(\lambda + \rho)P_1(t) + \lambda P_0(t),$$

$$\frac{dP_x(t)}{dt} = -(\lambda + \rho)P_x(t) + \lambda P_{x-1}(t) + \rho P_{x-2}(t) \text{ for } x \geq 2.$$

To investigate this system, take the Laplace transform (3, pp. 443-448) of $P_x(t)$, say $\mathcal{L}\{P_x(t)\}$, for the above probabilities. Then

$$\mathcal{L}\{P_0(t)\} = \frac{1}{\lambda + \rho + s},$$

$$\mathcal{L}\{P_1(t)\} = \frac{\lambda \mathcal{L}\{P_0(t)\}}{\lambda + \rho + s} = \frac{\lambda}{(\lambda + \rho + s)^2},$$

$$\mathcal{L}\{P_x(t)\} = \frac{\lambda \mathcal{L}\{P_{x-1}(t)\} + \rho \mathcal{L}\{P_{x-2}(t)\}}{\lambda + \rho + s} \text{ for } x \geq 2.$$

By repeated iteration one obtains

$$\mathcal{L}\{P_{2x}(t)\} = \sum_{j=0}^x \binom{2x-j}{j} \left[\frac{\lambda}{\lambda + \rho + s} \right]^{2(x-j)} \left[\frac{\rho}{\lambda + \rho + s} \right]^j \frac{1}{[\lambda + \rho + s]}$$

and

$$\mathcal{L}\{P_{2x+1}(t)\} = \sum_{j=0}^x \binom{2x-j+1}{j} \left[\frac{\lambda}{\lambda + \rho + s} \right]^{2(x-j)+1} \left[\frac{\rho}{\lambda + \rho + s} \right]^j \frac{1}{[\lambda + \rho + s]}.$$

This is the Laplace transformation with respect to t of the convolution of the distribution of Y , a univariate Poisson

with mean λt , and $2Z$, which has $\Pr \{2Z = 2z\} = \frac{e^{-\rho t} (\rho t)^z}{z!}$, $z = 0, 1, 2, \dots$, and all other probabilities zero. Verification can be made by direct computation of the Laplace transform for the distribution of $Y + 2Z$.

Therefore, a particular univariate multiple Poisson distribution has been derived as the distribution of cumulative counts of materializations up to fixed time for a Markov process arising from suitable postulates. This procedure can undoubtedly be extended to derive any multivariate multiple Poisson; however the details have not been investigated here.

X. CONCLUSIONS

Multivariate and multiple Poisson distributions are derived as limiting distributions in several situations. Supplementary results which either characterize these distributions or are required by the derivation procedures are also presented.

The first situation -- Chapters II and III -- pertains to general sequences of distributions converging to the multivariate Poisson and investigates their properties. It is shown by means of a special case of a theorem by Chernoff, proved here independently, that (1) sub-sums of random variables whose joint distribution converges to a multivariate independent Poisson converge to a multivariate Poisson, and (2) sub-sums of random variables whose joint distribution converges to a multivariate Poisson converge to a multivariate multiple Poisson. An example is constructed of a non-independent bivariate distribution with Poisson marginals such that $E[X_1 X_2] = E[X_1]E[X_2]$, $E[X_1^2 X_2] = E[X_1^2]E[X_2]$, $E[X_1 X_2^2] = E[X_1]E[X_2^2]$.

It is also shown that the multivariate Poisson distribution satisfies the multivariate Carleman uniqueness criterion, and thus, upon application of a theorem by Haviland, that a sequence of random variables whose moments converge to those of a multivariate Poisson, converges in distribution to that

same multivariate Poisson.

The second situation -- Chapters IV, V, VI, VII, VIII -- pertains to the derivation of the multivariate and multivariate multiple Poisson distributions as limiting distributions of numbers of runs (as usually defined) and other configurations on a circle. Under the condition of constant expectations, it is shown, by means of a generalization of a theorem of Fréchet, that the number of black and white ball runs on a circle is asymptotically multivariate independent Poisson. A corollary to this shows that the sum of numbers of runs of the same length is asymptotically univariate Poisson, a result analogous to that of von Mises. A second corollary states that the bivariate distribution of number of runs of black balls of length $k(n)$ and number of runs of both black and white balls of length $k(n)$ is a bivariate Poisson, called semi-Poisson in this thesis, which assigns positive mass to only one-half the positive quadrant.

When more general configurations are considered in the bivariate case, a correlated Poisson results.

It is shown that when configuration lengths are independent of n , the asymptotic multivariate distribution of general configurations involves independent, equivalent or constant Poisson random variables.

Finally, a theorem is proved which, using the property of infinite divisibility, states that all configurations whose

counts are essentially asymptotically marginally Poisson, must be multivariate Poisson. This shows that there cannot essentially exist a set of configurations whose counts are marginally Poisson but not multivariate Poisson.

The third situation considered -- Chapter IX -- is that of the derivation of a multiple Poisson from a suitably defined Markov process.

A possible application of the asymptotic results pertaining to simultaneous configuration counts lies in their providing approximate probabilities for the joint materializations of such counts. The setting down of postulates yielding univariate and multiple Poisson space distributions from discontinuous Markov processes provides some insight into the type of physical phenomena which might be expected to give rise to these distributions.

There are various possibilities for further work in this area, as suggested by this investigation. It is to be noted that the asymptotic independence of runs under the Poisson limit stands in contrast to the correlation obtained when the usual normal limit is taken. This suggests, as an interesting problem for further study, the investigation of the precise extent of the domain of attraction of the independent Poisson in configuration count problems of this type. In addition, whereas the present work has concentrated on arbitrary configurations formed by sampling from a binomial population, it

should be of interest to extend this research to cases of some of the additional dichotomy and indeed other polytomy criteria. Finally, it should be fairly simple to set forth a multivariate analogue to the Markov process which yields a univariate multiple Poisson.

XI. REFERENCES

1. Aitken, A.C. Statistical mathematics. New York, N.Y., Interscience Publishers, Inc. 1942.
2. Anderson, T.W. An introduction to multivariate statistical analysis. New York, John Wiley and Sons. 1958.
3. Bharucha-Reid, A.T. Elements of the theory of Markov processes and their applications. New York, McGraw-Hill Book Company, Inc. 1960.
4. Birkhoff, G. and MacLane, S. A survey of modern algebra. New York, Macmillan Co. 1941.
5. Bortkiewicz, L.v. Die iterationen. Berlin, Julius Springer. 1917.
6. Campbell, J.T. The Poisson correlation function. Proceedings of the Edinburgh Mathematical Society. Ser. 2, 4:18-76. 1934.
7. Chernoff, Herman. Lectures on order relations and convergence in distribution. Transcribed by John W. Pratt at Stanford University. (Mimeo.) Ames, Iowa, Department of Statistics, Iowa State University of Science and Technology. ca. 1957.
8. Cramér, Harald. Mathematical methods of statistics. Princeton, Princeton University Press. 1957.
9. Cramér, Harald and Wold, H. Some theorems on distribution functions. Journal London Mathematical Society. 11:290-294. 1936.
10. David, H.T. The sample mean among the intermediate order statistics, [To be published in Annals of Mathematical Statistics ca. September 1962].
11. Dwass, Meyer and Teicher, Henry. On infinitely divisible random vectors. Annals of Mathematical Statistics. 28:461-470. 1957.
12. Feller, William. An introduction to probability theory and its applications. 2nd ed. Vol. 1. New York, John Wiley and Sons, Inc. 1957.

13. Fréchet, Maurice. Les probabilités associées à un système d'événements compatibles et dépendants. *Actualités Scientifiques et Industrielles*. No. 859. 1940.
14. Fréchet, Maurice and Shohat, J. A proof of the generalized second limit theorem in the theory of probability. *Transactions of the American Mathematical Society*. 33:533-543. 1931.
15. Frisch, Ragnar. Sur les semi-invariants et moments employés dans l'étude des distributions statistiques. *Skrifter Utgitt av det Norske Videnskaps-Akademi: II. Hist.-Filos. Klasse*. No. 3. 1926.
16. Gurland, John. Lectures in advanced probability. (Mimeo.) Ames, Iowa, Department of Statistics, Iowa State University of Science and Technology. 1955.
17. Halmos, Paul R. Measure theory. New York, D. Van Nostrand Company, Inc. 1954.
18. Haviland, E.K. On the theory of absolutely additive distribution functions. *American Journal of Mathematics*. 56:625-658. 1934.
19. Iyer, P.V. Krishna. A theorem on factorial moments and its applications. *Annals of Mathematical Statistics*. 29:254-261. 1958.
20. Kendall, Maurice G. The advanced theory of statistics. 5th ed. Vol. 1. New York, Hafner Publishing Company. 1952.
21. Loève, Michel. On sets of probability laws and their limit elements. *University of California Publications in Statistics*. 1:53-88. 1950.
22. Mises, R. von. Das problem der iterationen. *Zeitschrift für angewandte mathematik und mechanik*. 1:298-307. 1921.
23. Mood, A.M. The distribution theory of runs. *Annals of Mathematical Statistics*. 11:367-392. 1940.
24. Mosteller, Frederick. Note on an application of runs to quality control charts. *Annals of Mathematical Statistics*. 12:228-232. 1941.

25. Richardson, C.H. The calculus of finite differences.
New York, D. Van Nostrand Company, Inc. 1954.
26. Shohat, J.A. and Tamarkin, J.D. The problem of moments.
New York, American Mathematical Society. 1943.
27. Teicher, Henry. On the factorization of distributions.
Annals of Mathematical Statistics. 25:769-774.
1954.
28. Teicher, Henry. On the multivariate Poisson distribution.
Skandinavisk Aktuarietidskrift. 37:1-9. 1954.
29. Thielman, Henry P. Theory of functions of real variables.
New York, Prentice-Hall, Inc. 1953.
30. Wald, A. and Wolfowitz, J. On a test whether two samples
are from the same population. Annals of Mathe-
matical Statistics. 11:147-162. 1940.
31. Wishart, J. and Hirschfeld, H.O. A theorem concerning
the distribution of joins between line segments.
London Mathematical Society Journal. 11:227-235.
1936.

XII. ACKNOWLEDGEMENTS

An honest expression of the indebtedness of the author to Dr. Herbert T. David for his imaginative guidance in this investigation would be embarrassing. It is enough to say that his acute observations, lavish gifts of time, and cheerful sense of humor cannot be sufficiently appreciated.

In addition, thanks are due to Dr. T. A. Bancroft for his assistance in the organization of the author's scholastic program at Iowa State University and for his help in securing the support of the National Institutes of Health for this research.